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# Abstract

## 摘要


The asymptotic safety approach to quantum gravity provides a possible solution to the problem of the quantization of the gravitational field. It is based on the existence of an ultraviolet non-Gaussian fixed point for the couplings of the $\sqrt{g}R$ and $\sqrt{g}$ operators of the Euclidean theory. The most compelling implication of this fact is a vanishing Newton’s constant $G$ at infinitely high energies. In this review the possible strategies to implement the details of this mechanism in cosmology are presented and the results are briefly discussed. In its simplest realization the universe emerges from a state of zero entropy end evolves towards an accelerated phase driven by a time-dependent Newton constant and cosmological constant $\Lambda$ according to the renormalization group for

量子引力渐近安全方法为引力场的量子化问题提供了一种可行解决方案。该方法基于欧几里得理论中 $\sqrt{g}R$ 和 $\sqrt{g}$ 耦合存在紫外非高斯不动点。这一结论最引人关注的推论是，牛顿常数 $G$ 在无限高能处趋于零。本综述介绍了将该机制细节应用于宇宙学的可行方案，并对相关结果进行了简要讨论。根据重整化群，该机制最简实现形式中，宇宙从零熵状态产生，随后演化为由含时牛顿常数和宇宙学常数 $\Lambda$ 驱动的加速阶段，


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A. Bonanno ( )

A. 博南诺 ( )

INAF, Osservatorio Astrofisico di Catania, Catania, Italy

意大利卡塔尼亚国家天体物理研究所卡塔尼亚天体物理天文台

INFN, Sezione di Catania, Catania, Italy

意大利卡塔尼亚国家核物理研究所卡塔尼亚分部

e-mail: alfio.bonanno@inaf.it

电子邮箱:alfio.bonanno@inaf.it


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quantum gravity. In this framework bouncing and non-singular universe models can be dynamically generated from the vanishing of the Newton’s constant at high energies. Important modifications of the original $R + R^2$ Starobinsky model are expected at the inflationary scale. Direct computations of the graviton spectral function can in principle provide the initial spectrum of gravitational waves in the early universe. At late times strong renormalization effects are expected at large cosmological scales that can drive the observed accelerated expansion of the universe. The new era of precision cosmology represents an important occasion to finally test the physical predictions of this new theory of quantum gravity.

量子引力。在该框架下，高能处牛顿常数趋于零可以动力学生成反弹非奇异宇宙模型。在暴胀尺度下，原始 $R+R^2$ 斯塔罗宾斯基模型预计会发生重要修改。原则上，引力子谱函数的直接计算可以给出早期宇宙引力波的初始谱。在晚期，大宇宙学尺度上预计会产生强重整化效应，该效应可以解释观测到的宇宙加速膨胀。精度宇宙学的新时代为最终检验这一全新量子引力理论的物理预言提供了重要契机。

## Keywords

## 关键词

Asymptotic safety - Theoretical cosmology - Renormalization group

渐近安全 - 理论宇宙学 - 重整化群

## Introduction

## 引言

The asymptotic safety approach to the quantization of the gravitational field has recently received much attention. It is based on the idea that, as long as we do not insist on the notion of continuum limit tailored to perturbation theory, gravity can be treated along the same lines of similar quantum field theories whose continuum limit is defined non-perturbatively. This possibility was first suggested by Weinberg [1] and further implemented by Reuter [2] by means of the Wilsonian renormalization group (RG) formalism in quantum field theory (QFT) [3-11]. (See the recent books [12,13] for a pedagogical introduction to the subject.)

引力场量子化的渐近安全方法近年来受到了广泛关注。该方法的核心思想是: 只要我们不坚持微扰论框架下的连续极限概念，就可以沿用其他非微扰定义连续极限的量子场论的思路来处理引力。这一可能性最早由温伯格 [1] 提出，随后罗伊特 [2] 借助量子场论 (QFT) 中的威尔逊重整化群 (RG) 框架对其进行了实现 [3-11]。(相关入门介绍可参见最新专著 [12,13])

It is possible to explain the technical mechanism which lies at the root of the non-perturbative renormalization of Einstein gravity in simple physical terms. Perhaps the most illuminating discussion in this context has been presented by Polyakov [14] who noticed that as gravity is always attractive, and thus, a larger cloud of virtual particle implies a stronger gravitational force, Newton’s constant $G$ should be antiscreened at small distances. The implication of this behavior is that the dimensionless coupling constant $g(\ell) = G(\ell)/\ell^2$ tends to a finite nonzero limit at small distances

我们可以用简单的物理语言解释爱因斯坦引力非微扰重整化的核心技术机制。在该方向上，波利亚科夫 [14] 给出了一个最具启发性的讨论: 他指出，由于引力始终是吸引力，更大的虚粒子云意味着更强的引力作用，因此牛顿常数 $G$ 在小尺度下会表现为反屏蔽效应。这一行为的推论是，无量纲耦合常数 $g(\ell) = G(\ell)/\ell^2$ 在小距离下会趋向一个非零有限极限

$$\lim_{\ell \to 0} g(\ell) \propto g^* \neq 0 \tag{1}$$

as $G$ scales as $\ell^2$ according to its natural dimensions. A theory whose dimensionless coupling constant approaches a non-Gaussian (nonvanishing) fixed point (NGFP) in the short distance limit as in (1) is called asymptotically safe (at variance with the more familiar case of the asymptotic freedom where $g^* = 0$ .) While the behavior in (1) has been conjectured in [14] on the basis of a similarity with the nonlinear $\sigma$ -model, rigorous calculations based on the application of the non-perturbative flow equation have indeed shown that the ultraviolet critical manifold of a Lagrangian of the type

，这是因为 $G$ 依照自身自然量纲会按照 $\ell^2$ 的形式标度。如果一个理论的无量纲耦合常数在短距离极限下如 (1) 式那样趋近于非高斯 (非零) 不动点 (NGFP)，该理论就被称为渐近安全的 (这和我们更熟悉的渐近自由情况不同，渐近自由中 $g^* = 0$ )。尽管 (1) 式的行为是文献 [14] 基于和非线性 $\sigma$ 模型的相似性提出的猜想，但基于非微扰流方程的严格计算确实表明，此类拉格朗日量的紫外临界流形

$$\mathcal{L}_{\text{eff}} = \sum_i g_i R^i \tag{2}$$

is controlled by a NGFP [15, 16]. In particular, it turns out that the coupling constants $g_i$ are all scale-dependent albeit only a finite number of those are needed to control the continuum limit of the theory. On the other hand, the NGFP cannot be approached in perturbation theory and non-perturbative approaches, like the functional renormalization group equation (FRGE), must be implemented [17, 18].

由非高斯不动点 NGFP 控制 [15,16]。具体来说，我们发现耦合常数 $g_i$ 都具有标度依赖性，但只需要有限个耦合常数就可以控制理论的连续极限。另一方面，非高斯不动点无法在微扰论中得到，必须借助如泛函重整化群方程 (FRGE) 这类非微扰方法进行研究 [17,18]

In fact FRGE is capable to reproduce the running of $G$ and the cosmological constant $\Lambda$ at various energy scales and has shed light on the properties of the quantum structure of the space-time near the Planck scale. A number of approaches to implement the AS framework in cosmology using FRGE have thus recently appeared. More specifically, we can distinguish the following main strategies:

事实上，泛函重整化群方程 FRGE 能够重现不同能标下 $G$ 和宇宙学常数 $\Lambda$ 的跑动行为，并且揭示了普朗克尺度附近时空量子结构的性质。近年来，因此涌现了许多借助 FRGE 在宇宙学中实现渐近安全 AS 框架的研究。具体来说，主要可以分为以下几类研究策略:

- Direct implementation of the antiscreening character of $G$ in the field equations from the $\beta$ -functions from FRGE.

- 在来自 FRGE 的 $\beta$ 函数给出的场方程中直接引入 $G$ 的反屏蔽性质。

- Inflationary models based on the structure of the effective Lagrangian as emerging from the NGFP.

- 基于非高斯不动点导出的有效拉格朗日量结构构建的暴涨模型。

- Non-perturbative calculation of the spectral properties of the graviton propagator in the flow equation.

- 在流方程框架下对引力子传播子的谱性质进行非微扰计算。

In this work the basic assumptions of the above approaches will be reviewed and discussed.

本文将回顾并讨论上述各类研究方法的基本假设。

## Wilsonian Action in Cosmology

## 宇宙学中的威尔逊作用量

Let us now briefly review the basic idea of the non-perturbative renormalization and its application in gravity. The Wilsonian RG transformation [19], as opposed to the more conventional RG transformation based on the rescaling properties of Green's functions, represents a systematic method to construct a low-energy, effective theory, from a fundamental one defined at the cutoff scale.

现在我们简要回顾非微扰重整化的基本思想及其在引力中的应用。与基于格林函数标度性质的传统重整化群 (RG) 变换不同，威尔逊 RG 变换 [19] 是一套从定义在截断能标处的基础理论出发，构建低能有效理论的系统方法。

Let us imagine the theory being described by a collection of fields $\varphi$ whose properties are encoded in an action $S[\varphi]$ and assume that the following expansion in terms of local field operator holds:

我们假设该理论由一组场 $\varphi$ 描述，场的性质由作用量 $S[\varphi]$ 刻画，并且满足如下局域场算符展开:

$$S[\phi] = \sum_i g_i(\Lambda)\, s_i[\phi] \tag{3}$$

We would like to describe the same system with a new set of field operators $\Phi$ that are obtained by averaging the original field variables $\varphi$ on a larger volume $\Omega \sim 1/k^d$ where $k$ is a low-energy cutoff. The actual averaging procedure is performed by an average operator $C_\Omega(\varphi)$ (See [20] for an explicit construction of $C_\Omega$ in the case of maximally symmetric spaces.). The requirement that the partition function remains invariant under this transformation defines the Wilsonian (also called blocked) action $S_\Omega(\varphi)$ as

我们希望用一组新的场算符 $\Phi$ 描述同一系统，新场是对原场变量 $\varphi$ 在更大体积 $\Omega \sim 1/k^d$ 上做平均得到的，其中 $k$ 是低能截断。具体的平均过程由平均算符 $C_\Omega(\varphi)$ 完成 (最大对称空间中 $C_\Omega$ 的显式构造可参见文献 [20])。配分函数在该变换下保持不变的要求定义了威尔逊作用量 (也称为分块作用量) $S_\Omega(\varphi)$，形式为:

$$e^{-S_\Omega[\Phi]} = \int \mathcal{D}\varphi \prod_x \delta\left(\Phi(x) - C_\Omega(\varphi)(x)\right) e^{-S[(\varphi)]} \tag{4}$$

The $\delta$ -constraint in (4) can equivalently be implemented in the momentum space and in this case the Wilsonian action at the scale $S_k[\Phi]$ is obtained by means of a functional integration on the fast degrees of freedom,

(4) 式中的 $\delta$ 约束也可以等价地在动量空间中实现，此时能标 $S_k[\Phi]$ 处的威尔逊作用量可以通过对快自由度做泛函积分得到，

$$e^{-S_k[\Phi]} = \int \mathcal{D}\zeta e^{-S[\zeta+\Phi]} \tag{5}$$

where $\zeta$ represent the (high-frequency) components of the original field $\varphi$ , the ones with momenta $p$ greater than some infrared cutoff $k$ , and $\Phi$ the low-frequency field with momenta $p \leq k$ . It is possible to show that in the limit $k \to 0$ the Wilsonian action reduces to the effective potential, i.e., the nonderivative part of the effective action, but in general for $k \neq 0$ it represents the action for the coarse-grained field, i.e., the average of the original field on regions of typical size $\ell \sim 1/k$ . The computation of $S_k$ is obtained by means of the functional renormalization group equation which in principle can handle the infinitely many new coupling constant generated by the blocking procedure [18]. The theory is predictive when only a finite number of these new effective interactions are needed to perform the continuum limit.

其中 $\zeta$ 是原场 $\varphi$ 的 (高频) 分量，也就是动量 $p$ 大于某红外截断 $k$ 的分量，而 $\Phi$ 是动量为 $p \leq k$ 的低频场。可以证明，在极限 $k \to 0$ 下，威尔逊作用量退化为有效势，也就是有效作用量的非导数部分；但一般情况下，对于 $k \neq 0$ ，它描述粗粒化场的作用量，即原场在典型尺寸 $\ell \sim 1/k$ 区域上的平均。$S_k$ 可通过泛函重整化群方程计算得到，该方程原则上可以处理分块过程产生的无穷多个新耦合常数 [18]。当只需要有限个这类新有效相互作用就可以取连续极限时，该理论是可预言的。

The FRGE for the gravitational effective average action $\Gamma_k$ has been obtained first by [2]

引力有效平均作用量 $\Gamma_k$ 的泛函重整化群方程 (FRGE) 最早由文献 [2] 得到

$$\partial_k \Gamma_k[g, \bar{g}] = \frac{1}{2} \mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \partial_k \mathcal{R}_k\right]. \tag{6}$$

where $\Gamma_k$ is a functional obtained within the background field method, by splitting the metric $g_{\mu\nu}$ into a fixed background $\bar{g}_{\mu\nu}$ and fluctuations $h_{\mu\nu}$ . The Hessian $\Gamma_k^{(2)}$ is the second functional derivative of $\Gamma_k$ with respect to the fluctuation field at a fixed background and $\mathcal{R}_k$ provides a scale-dependent mass term for fluctuations with momenta $p^2 \ll k^2$ with the RG scale $k$ constructed from the background metric. The interplay of $\mathcal{R}_k$ in the numerator and denominator renders the trace both infrared and ultraviolet finite and ensures that the flow of $\Gamma_k$ is actually governed by fluctuations with momentum $p^2 \approx k^2$ . In this sense, the flow equation realizes Wilson's idea of renormalization by integrating out "short-scale fluctuations" with momenta $p^2 \ll k^2$ such that $\Gamma_k$ provides an effective description of physics for typical scales $k^2$ .

其中 $\Gamma_k$ 是背景场方法中得到的泛函，该方法将度规 $g_{\mu\nu}$ 拆分为固定背景 $\bar{g}_{\mu\nu}$ 和涨落 $h_{\mu\nu}$ 。黑塞矩阵 $\Gamma_k^{(2)}$ 是固定背景下涨落场关于 $\Gamma_k$ 的二阶泛函导数，$\mathcal{R}_k$ 为动量为 $p^2 \ll k^2$ 的涨落提供了依赖能标的质量项，RG 能标 $k$ 由背景度规构造。分子和分母中的 $\mathcal{R}_k$ 共同作用使得迹在红外和紫外都是有限的，并保证 $\Gamma_k$ 的流确实由动量为 $p^2 \approx k^2$ 的涨落主导。在此意义下，该流方程实现了威尔逊的重整化思想: 通过积掉动量满足 $p^2 \ll k^2$ 的“小尺度涨落”，让 $\Gamma_k$ 给出典型尺度 $k^2$ 下物理的有效描述。

The simplest approximation of the gravitational RG flow is obtained from projecting the FRGE onto the Einstein-Hilbert action approximating $\Gamma_k$ by

引力 RG 流的最简单近似，是将 FRGE 投影到爱因斯坦-希尔伯特作用量，将 $\Gamma_k$ 近似为

$$S_k = \frac{1}{16\pi G_k}\int d^4x\sqrt{g}\left[-R+2\Lambda_k\right] + \text{ gauge-fixing and ghost terms }. \tag{7}$$

This ansatz comprises two scale-dependent coupling constants, Newton's constant $G_k$ and a cosmological constant $\Lambda_k$ . The scale dependence of these couplings is conveniently expressed in terms of their dimensionless counterparts

该假设包含两个依赖能标的耦合常数: 牛顿常数 $G_k$ 和宇宙学常数 $\Lambda_k$ 。这些耦合的能标依赖可以方便地用它们的无量纲形式表示

$$\lambda_k \equiv k^{-2}\Lambda_k,\ g_k \equiv k^2 G_k, \tag{8}$$

and captured by the beta functions

并由 β 函数刻画

$$k\partial_k g_k = \beta_g\left(g_k,\lambda_k\right),\ k\partial_k\lambda_k = \beta_\lambda\left(g_k,\lambda_k\right). \tag{9}$$

Evaluating the beta functions [2] for the so-called optimized regulator [21] gives

对所谓优化调节子 [21] 计算 β 函数 [2]，得到

$$\beta_\lambda = \left(\eta_N-2\right)\lambda + \frac{g}{12\pi}\left[\frac{30}{1-2\lambda}-24-\frac{5}{1-2\lambda}\eta_N\right] \tag{10}$$

$$\beta_g = \left(2+\eta_N\right)g$$

with the anomalous dimension of Newton’s constant $\eta_N \equiv \left(G_k\right)^{-1}k\partial_k G_k$ being given by

其中牛顿常数的反常维度 $\eta_N \equiv \left(G_k\right)^{-1}k\partial_k G_k$ 由下式给出

$$\eta_N = \frac{gB_1(\lambda)}{1-gB_2(\lambda)} \tag{11}$$

where

其中

$$B_1(\lambda) = \frac{1}{3\pi}\left[\frac{5}{1-2\lambda}-\frac{9}{(1-2\lambda)^2}-7\right],\ B_2(\lambda) = -\frac{1}{12\pi}\left[\frac{5}{1-2\lambda}-\frac{6}{(1-2\lambda)^2}\right]. \tag{12}$$

The beta functions (10) encode the scale dependence of the dimensionless Newton's constant and cosmological constant. In particular, they contain the information on fixed points $g_*$ of the RG flow where, by definition of a fixed point, the beta functions vanish simultaneously, $\beta^a\left(g^a\right)\big|_{g^a=g^a_*} = 0$ constants. In the vicinity of a fixed point, the properties of the RG flow are captured by linearizing the beta functions around the fixed point. Defining the stability matrix $\mathbf{B}_{ab} \equiv \partial_{g^b}\beta_{g^a}\big|_{g=g_*}$ , the linearized flow takes the form

式 (10) 的 β 函数编码了无量纲牛顿常数与宇宙常数的标度依赖关系。具体而言，它们包含了 RG 流不动点 $g_*$ 的信息: 根据不动点的定义，此时 β 函数同时为零，$\beta^a\left(g^a\right)|_{g^a=g^a_*} = 0$ 为常数。在不动点附近，RG 流的性质可通过将 β 函数在不动点处线性化得到。定义稳定性矩阵 $\mathbf{B}_{ab} \equiv \partial_{g^b}\beta_{g^a}\big|_{g=g_*}$ 后，线性化流形式为

$$g^a(k) = g^a_* + \sum_I C^I V^a_I \left(\frac{k_0}{k}\right)^{\theta_I}. \tag{13}$$

Here the $V_I$ denote the right eigenvectors of $\mathbf{B}$ with eigenvalues $-\theta_I$ such that $\sum_b \mathbf{B}_{ab} V^b_I = -\theta_I V^a_I, k_0$ is a fixed reference scale and the $C^I$ are constants of integration. If $\mathrm{Re}\,\theta_I > 0$ , the flow along the eigendirection $V_I$ automatically approaches the fixed point $g^a_*$ as $k \to \infty$ . In this case, the $C_I$ has a status of a free parameter. Analogously, eigendirections with $\mathrm{Re}\,\theta_I < 0$ are repelled from the fixed point as $k \to \infty$ . The requirement that the fixed point controls the flow at high energy then demands that the corresponding integration constants $C_I$ must be set to zero. Compared to the effective field theory framework, asymptotic safety then potentially fixes an infinite number of free couplings, leading to a vast increase in predictive power.

此处 $V_I$ 表示 $\mathbf{B}$ 的右特征向量，对应的特征值为 $-\theta_I$ ，其中 $\sum_b \mathbf{B}_{ab} V^b_I = -\theta_I V^a_I, k_0$ 是固定参考标度，$C^I$ 为积分常数。若 $\mathrm{Re}\,\theta_I > 0$ ，当 $k \to \infty$ 时，沿特征方向 $V_I$ 的流动会自动趋近不动点 $g^a_*$ 。这种情况下，$C_I$ 具有自由参数的性质。类似地，当 $k \to \infty$ 时，满足 $\mathrm{Re}\,\theta_I < 0$ 的特征方向会被不动点排斥。若要求不动点控制高能区的流动，则对应的积分常数 $C_I$ 必须置零。与有效场论框架相比，渐近安全由此可以固定无穷多个自由耦合，极大提升了预测能力。

The beta functions (10) give rise to two fixed points. Firstly, the Gaussian fixed point (GFP) is situated at $(g_*, \lambda_*) = (0, 0)$ . It corresponds to a free theory where the stability coefficients are determined by the mass dimension of the coupling constant. Thus, the GFP is a saddle point in the $g - \lambda$ -plane: linearized solutions with $g > 0$ are repelled from this fixed point for $k \to \infty$ . This feature reflects the perturbative non-renormalizability of the Einstein-Hilbert action in the Wilsonian language.

式 (10) 的 β 函数给出两个不动点。首先，高斯不动点 (GFP) 位于 $(g_*, \lambda_*) = (0, 0)$ 。它对应自由理论，其稳定性系数由耦合常数的质量维度决定。因此，高斯不动点是 $g - \lambda$ 平面上的鞍点: 当 $k \to \infty$ 时，满足 $g > 0$ 的线性化解会被该不动点排斥。这一特征在威尔逊语言中反映了爱因斯坦-希尔伯特作用量的微扰不可重整性。

In addition, the flow possesses a non-Gaussian fixed point (NGFP) located at

此外，该流动存在一个非高斯不动点 (NGFP)，位于

$$g_* = 0.707, \ \lambda_* = 0.193. \tag{14}$$

From Eq. (10) one sees that the anomalous dimension of Newton's constant at this fixed point is $\eta_N = -2$ . Its stability coefficients are given by

从式 (10) 可以看出，该不动点处牛顿常数的反常维度为 $\eta_N = -2$，其稳定性系数由下式给出

$$\theta_{1,2} = 1.48 \pm 3.04i \tag{15}$$

such that RG flows in its vicinity actually spiral into the fixed point as $k \to \infty$. In the fixed-point regime, (14) then entails that the dimensionful coupling constants scale according to

使得当 $k \to \infty$ 时，其邻域内的 RG 流实际上会螺旋收敛到该不动点。在不动点区域，式 (14) 可推出量纲耦合常数满足如下标度关系

$$\lim_{k\to\infty} G_k = g_* k^{-2},\ \lim_{k\to\infty} \Lambda_k = \lambda_* k^2. \tag{16}$$

In particular the dimensionful Newton’s constant vanishes as $k \to \infty$, entailing that the asymptotic safety mechanism renders gravity antiscreening. At last the field equations can be obtained from

具体而言，量纲牛顿常数随 $k \to \infty$ 趋于零，这意味着渐近安全机制使引力成为反屏蔽效应。最终场方程可由下式得到

$$\frac{\delta S_k}{\delta g_{\mu\nu}(x)} \left[\langle g\rangle_k\right] = 0 \tag{17}$$

so that we can write the following scale-dependent field equations

因此我们可以写出如下依赖标度的场方程

$$R_{\mu\nu}\left[\langle g\rangle_k\right] - \frac{1}{2} R\left[\langle g\rangle_k\right]\langle g_{\mu\nu}\rangle_k = \Lambda(k)\langle g_{\mu\nu}\rangle_k + 8\pi G(k)\langle T_{\mu\nu}\rangle_k \tag{18}$$

where the evolution of $G(k)$ and $\Lambda(k)$ is determined by the $\beta$ -functions (10). Equation (18) coupled with the $\beta$ -functions can be used to describe the properties of classical solutions of the field equations ”dressed” at different energy scale $k$, a procedure called renormalization group improvement. This is, in fact, a standard device in particle physics in order to add the dominant quantum corrections to the Born approximation of a scattering cross section, for instance. In QED one starts from the classical potential energy $V_{\text{cl}}(r) = e^2/4\pi r$ and replaces $e^2$ by the running gauge coupling in the one-loop approximation:

其中 $G(k)$ 和 $\Lambda(k)$ 的演化由 $\beta$ 函数 (10) 确定。方程 (18) 与 $\beta$ 函数结合，可用于描述场方程的经典解在不同能标 $k$ 下“修饰”后的性质，这一过程称为重整化群改进。实际上，这是粒子物理中的标准方法，例如，用于给散射截面的玻恩近似添加主导量子修正。在量子电动力学中，我们从经典势能 $V_{\text{cl}}(r) = e^2/4\pi r$ 出发，将 $e^2$ 替换为单圈近似下的跑动规范耦合:

$$e^2(k) = e^2(k_0)\left[1 - b\ln(k/k_0)\right]^{-1},\ b \equiv e^2(k_0)/6\pi^2. \tag{19}$$

The crucial step is to identify the renormalization point $k$ with the inverse of the distance $r$ (this is possible because in the massless theory $r$ is the only dimensional quantity which can define a scale) so that the result of this substitution reads

关键步骤是将重整化点 $k$ 等同于距离 $r$ 的倒数 (这是可行的，因为在无质量理论中 $r$ 是唯一能定义能标的量纲量)，因此替换后的结果为

$$V(r) = -e^2\left(r_0^{-1}\right)\left[1 + b\ln(r_0/r) + O\left(e^4\right)\right]/4\pi r \tag{20}$$

where the IR reference scale $r_0 \equiv 1/k_0$ has to be kept finite to avoid IR divergences. It should be stressed that Eq. (20) is the correct (one-loop, massless) Uehling potential which is usually derived by more conventional perturbative methods. Obviously, the position-dependent renormalization group improvement $e^2 \to e^2(k)$ , $k \propto 1/r$ encapsulates the most important effects which the quantum fluctuations have on the electric field produced by a point charge. In gravity one can expect the same procedure works at least qualitatively [10]. On the other hand, when we promote $k$ to become a dynamical quantity at the level of the field equations, we impose a stronger mathematical compatibility between the running from the renormalization group and the field equation. Albeit this procedure is only self-consistent, the resulting cosmic dynamics lead to interesting phenomenological consequences.

其中红外参考标度 $r_0 \equiv 1/k_0$ 必须保持有限，以避免红外发散。需要强调的是，式 (20) 就是正确的 (单圈、无质量)Uehling 势，它通常通过更常规的微扰方法推导得到。显然，依赖位置的重整化群改进 $e^2 \to e^2(k)$、$k \propto 1/r$ 概括了量子涨落对点电荷产生电场最重要的效应。在引力中，我们可以预期相同的过程至少在定性层面成立 [10]。另一方面，当我们在场方程层面将 $k$ 提升为动力学量时，我们要求重整化群跑动和场方程之间满足更强的数学相容性。尽管该过程仅要求自洽，得到的宇宙动力学仍会产生有趣的唯象学结果。

## Homogeneous Cosmologies

## 均匀宇宙学

The application of the RG improvement level of the field equations to cosmology is straightforward in case of homogeneous cosmologies as in this case every physical quantity can only be a function of the cosmic time $t$ . In this case a time-dependent scale dependence of the type $G(t) \equiv G(k(t))$ and $\Lambda(t) \equiv \Lambda(k(t))$ into the Einstein equations is introduced

场方程的重整化群改进应用到宇宙学中，在均匀宇宙学场景下十分直接，因为该情形中所有物理量都仅能是宇宙时间 $t$ 的函数。在这种情况下，$G(t) \equiv G(k(t))$ 型和 $\Lambda(t) \equiv \Lambda(k(t))$ 型的依赖尺度的时间相关性被引入爱因斯坦方程中

$$R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R = -\Lambda(t)\,g_{\mu\upsilon} + 8\pi G(t)\,T_{\mu\upsilon} \tag{21}$$

so that the solutions are dynamically dressed. Let us assume $g_{\mu\nu}$ to describe a spatially flat Robertson-Walker metric with scale factor $a(t)$ and take $T_\mu{}^\upsilon =$ diag $[-\rho, p, p, p]$ to be the energy-momentum tensor of an ideal fluid with equation of state $p = w\rho$ where $w > -1$ is constant. Then the improved Einstein equation reduces to the modified Friedmann equation and a continuity equation:

因此解会被动力学修正。我们假设 $g_{\mu\nu}$ 描述空间平坦的罗伯逊-沃克度规，其尺度因子为 $a(t)$，并取 $T_\mu{}^\nu = \text{diag}[-\rho, p, p, p]$ 为理想流体的能量动量张量，物态方程为 $p = w\rho$，其中 $w > -1$ 为常数。随后改进的爱因斯坦方程可约化为修正弗里德曼方程与连续性方程:

$$H^2 = \frac{8\pi}{3} G(t)\rho + \frac{1}{3}\Lambda(t) \tag{22}$$

$$\dot{\rho} + 3H(\rho + p) = -\frac{\dot{\Lambda} + 8\pi\rho\dot{G}}{8\pi G}. \tag{23}$$

An integrability condition for the improved Einstein equation implied by Bianchi's identity $D^\mu\left[-\Lambda(t) g_{\mu\nu} + 8\pi G(t) T_{\mu\nu}\right] = 0$ must hold. It describes the energy exchange between the matter and gravitational degrees of freedom (geometry). One can then obtain $G(k)$ and $\Lambda(k)$ by solving the flow equation in the Einstein-Hilbert truncation with a sharp cutoff once a cutoff identification is made. For actual calculations it is natural to employ the cutoff identification

由比安基恒等式 $D^\mu\left[-\Lambda(t) g_{\mu\nu} + 8\pi G(t) T_{\mu\nu}\right] = 0$ 给出的改进爱因斯坦方程的可积性条件必须成立，它描述了物质与引力自由度 (几何) 之间的能量交换。一旦选定截断识别，通过求解带尖锐截断的爱因斯坦-希尔伯特截断中的流方程，就可以得到 $G(k)$ 和 $\Lambda(k)$。对于实际计算，自然采用下述截断识别

$$k(t) = \xi H(t) \tag{24}$$

where $\xi$ is a fixed positive constant of order unity. As long as $H > 0$, this is a preferred choice since in a Robertson-Walker geometry the Hubble parameter measures the curvature of space-time; its inverse $H^{-1}$ defines the size of the "Einstein elevator." From a more formal point of view, the Ricci curvature $R$ of the background space-time acts as a mass term in the fluctuation determinant in the gravity sector of the FRGE for gravity. If no other scales are present in the system, for a maximally symmetric space-time, (24) is therefore a reasonable choice.

其中 $\xi$ 是量级为 1 的固定正的常数。只要满足 $H > 0$，这就是一个优选选择，因为在罗伯逊-沃克几何中，哈勃参数度量了时空的曲率；其倒数 $H^{-1}$ 定义了“爱因斯坦电梯”的尺度。从更形式化的角度来看，背景时空的里奇曲率 $R$ 在引力的泛函重整化群方程的引力部分中，作为涨落行列式的质量项存在。若系统中不存在其他标度，那么对于最大对称时空，式 (24) 因此是一个合理的选择。

It is interesting to see how trajectories of the Einstein-Hilbert truncation can be matched against the observational data. This analysis is fairly robust and clear-cut; it does not involve the NGFP. All that is needed is the RG flow linearized about the Gaussian fixed point (GFP) located at $g = \lambda = 0$. In its vicinity one has $\Lambda(k) = \Lambda_0 + \upsilon\bar{G}k^4 + \cdots$ and $G(k) = \bar{G} + \cdots$. Or, in terms of the dimensionless couplings, $\lambda(k) = \Lambda_0/k^2 + \upsilon\bar{G}k^2 + \cdots, g(k) = \bar{G}k^2 + \cdots$. In the linear regime of the GFP, $\Lambda$ displays a running $\propto k^4$ and $G$ is approximately constant. Here $\upsilon$ is a positive constant of order unity [22]. These equations are valid if $\lambda(k) \ll 1$ and $g(k) \ll 1$. They describe a two-parameter family of RG trajectories labeled by the pair $\left(\Lambda_0, \bar{G}\right)$. It is useful to use an alternative labeling $(\lambda_\mathrm{T}, k_\mathrm{T})$ with $\lambda_\mathrm{T} \equiv \left(4\upsilon\Lambda_0\bar{G}\right)^{1/2}$ and $k_\mathrm{T} \equiv \left(\Lambda_0/\upsilon\bar{G}\right)^{1/4}$. The old labels are expressed in terms of the new ones as $\Lambda_0 = \frac{1}{2}\lambda_\mathrm{T}k_\mathrm{T}^2$ and $\bar{G} = \lambda_\mathrm{T}/2\upsilon k_\mathrm{T}^2$. It is furthermore convenient to introduce the abbreviation $g_\mathrm{T} \equiv \lambda_\mathrm{T}/2\upsilon$. When parameterized by the pair $(\lambda_\mathrm{T}, k_\mathrm{T})$, the trajectories assume the form

探究爱因斯坦-希尔伯特截断的轨迹如何与观测数据匹配是很有意义的。该分析相当稳健且结论明确；它不涉及 NGFP。我们仅需要对位于 $g=\lambda=0$ 处高斯不动点 (GFP) 做线性化处理后的 RG 流。在其邻域内有 $\Lambda(k)=\Lambda_0+\upsilon\bar{G}k^4+\cdots$ 和 $G(k)=\bar{G}+\cdots$ 。或者，用无量纲耦合表示为 $\lambda(k)=\Lambda_0/k^2+\upsilon\bar{G}k^2+\cdots, g(k)=\bar{G}k^2+\cdots$ 。在 GFP 的线性区间内，$\Lambda$ 呈现出跑动的 $\propto k^4$ ，且 $G$ 近似为常数。此处 $\upsilon$ 是一个量级为 1 的正常数 [22]。这些方程在 $\lambda(k)\ll 1$ 和 $g(k)\ll 1$ 的条件下成立。它们描述了由 $(\Lambda_0,\bar{G})$ 对标记的双参数 RG 轨迹族。使用替代标记 $(\lambda_{\mathrm{T}},k_{\mathrm{T}})$ 会更方便，其中 $\lambda_{\mathrm{T}}\equiv\left(4\upsilon\Lambda_0\bar{G}\right)^{1/2}$ 且 $k_{\mathrm{T}}\equiv\left(\Lambda_0/\upsilon\bar{G}\right)^{1/4}$ 。旧标记可以用新标记表示为 $\Lambda_0=\frac{1}{2}\lambda_{\mathrm{T}}k_{\mathrm{T}}^2$ 和 $\bar{G}=\lambda_{\mathrm{T}}/2\upsilon k_{\mathrm{T}}^2$ 。此外，引入缩写 $g_{\mathrm{T}}\equiv\lambda_{\mathrm{T}}/2\upsilon$ 会更方便。当用 $(\lambda_{\mathrm{T}},k_{\mathrm{T}})$ 对参数化时，轨迹形式为

$$\Lambda(k)=\frac{1}{2}\lambda_{\mathrm{T}}k_{\mathrm{T}}^2\left[1+(k/k_{\mathrm{T}})^4\right]\equiv\Lambda_0\left[1+(k/k_{\mathrm{T}})^4\right]\tag{25}$$

$$G(k)=\frac{\lambda_{\mathrm{T}}}{2\upsilon k_{\mathrm{T}}^2}\equiv\frac{g_{\mathrm{T}}}{k_{\mathrm{T}}^2}$$

or, in dimensionless form,

或者，以无量纲形式表示为

$$\lambda(k)=\frac{1}{2}\lambda_{\mathrm{T}}\left[\left(\frac{k_{\mathrm{T}}}{k}\right)^2+\left(\frac{k}{k_{\mathrm{T}}}\right)^2\right],\ g(k)=g_{\mathrm{T}}\left(\frac{k}{k_{\mathrm{T}}}\right)^2\tag{26}$$

As for the interpretation of the new variables, it is clear that $\lambda_{\mathrm{T}}\equiv\lambda(k\equiv k_{\mathrm{T}})$ and $g_{\mathrm{T}}\equiv g(k=k_{\mathrm{T}})$ , while $k_{\mathrm{T}}$ is the scale at which $\beta_\lambda$ (but not $\beta_g$ ) vanishes according to the linearized running: $\beta_\lambda(k_{\mathrm{T}})\equiv kd\lambda(k)/dk|_{k=k_{\mathrm{T}}}=0$ . Thus, we see that $(g_{\mathrm{T}},\lambda_{\mathrm{T}})$ are the coordinates of the turning point T in Fig. 1 and $k_{\mathrm{T}}$ is the scale at which it is passed. Let us now hypothesize that, within a certain range of $k$ -values, the RG trajectory realized in nature can be approximated by (26). In order to determine its parameters $\left(\Lambda_0,\bar{G}\right)$ or $(\lambda_{\mathrm{T}},k_{\mathrm{T}})$ , we must perform a measurement of $G$ and $\Lambda$ . If we interpret the observed values $G_{\text{observed}}=m_{\mathrm{Pl}}^{-2}, m_{\mathrm{Pl}}\approx 1.2\times 10^{19}\mathrm{GeV}$ , and $\Lambda_{\text{observed}}=3\Omega_{\Lambda 0}H_0^2\approx 10^{-120}m_{\mathrm{Pl}}^2$ as the running $G(k)$ and $\Lambda(k)$ evaluated at a scale $k\ll k_{\mathrm{T}}$ , then we get from (25) that $\Lambda_0=\Lambda_{\text{observed}}$ and $\bar{G}=G_{\text{observed}}$ . Using the definitions of $\lambda_{\mathrm{T}}$ and $k_{\mathrm{T}}$ along with $\upsilon=O(1)$ , this leads to the order-of-magnitude estimates $g_{\mathrm{T}}\approx\lambda_{\mathrm{T}}\approx 10^{-60}$ and $k_{\mathrm{T}}\approx 10^{-30}m_{\mathrm{Pl}}\approx\left(10^{-3}\ \mathrm{cm}\right)^{-1}$ . Because of the tiny values of $g_{\mathrm{T}}$ and $\lambda_{\mathrm{T}}$ , the turning point lies in the linear regime of the GFP.

至于新变量的物理诠释，显然有 $\lambda_{\mathrm{T}}\equiv\lambda(k\equiv k_{\mathrm{T}})$ 和 $g_{\mathrm{T}}\equiv g(k=k_{\mathrm{T}})$ ，而根据线性化跑动，$k_{\mathrm{T}}$ 是使 $\beta_\lambda$ (而非 $\beta_g$ ) 变为零的标度: $\beta_\lambda(k_{\mathrm{T}})\equiv kd\lambda(k)/dk|_{k=k_{\mathrm{T}}}=0$ 。由此可知，$(g_{\mathrm{T}},\lambda_{\mathrm{T}})$ 是图 1 中拐点 T 的坐标，$k_{\mathrm{T}}$ 则是经过该拐点的标度。现在我们假设，在 $k$ 值的一定范围内，自然界实现的重整化群轨迹可以用 (26) 式近似。为了确定其参数 $\left(\Lambda_0,\bar{G}\right)$ 或 $(\lambda_{\mathrm{T}},k_{\mathrm{T}})$ ，我们必须对 $G$ 和 $\Lambda$ 进行测量。若我们将观测值 $G_{\text{observed}}=m_{\mathrm{Pl}}^{-2}, m_{\mathrm{Pl}}\approx 1.2\times 10^{19}\mathrm{GeV}$ 和 $\Lambda_{\text{observed}}=3\Omega_{\Lambda 0}H_0^2\approx 10^{-120}m_{\mathrm{Pl}}^2$ 诠释为标度 $k\ll k_{\mathrm{T}}$ 处跑动的 $G(k)$ 和 $\Lambda(k)$ ，那么我们由 (25) 式可得 $\Lambda_0=\Lambda_{\text{observed}}$ 和 $\bar{G}=G_{\text{observed}}$ 。结合 $\lambda_{\mathrm{T}}$ 和 $k_{\mathrm{T}}$ 的定义以及 $\upsilon=O(1)$ ，可得到数量级估计 $g_{\mathrm{T}}\approx\lambda_{\mathrm{T}}\approx 10^{-60}$ 和 $k_{\mathrm{T}}\approx 10^{-30}m_{\mathrm{Pl}}\approx\left(10^{-3}\ \mathrm{cm}\right)^{-1}$ 。由于 $g_{\mathrm{T}}$ 和 $\lambda_{\mathrm{T}}$ 数值极小，拐点位于高斯固定点 (GFP) 的线性区域。

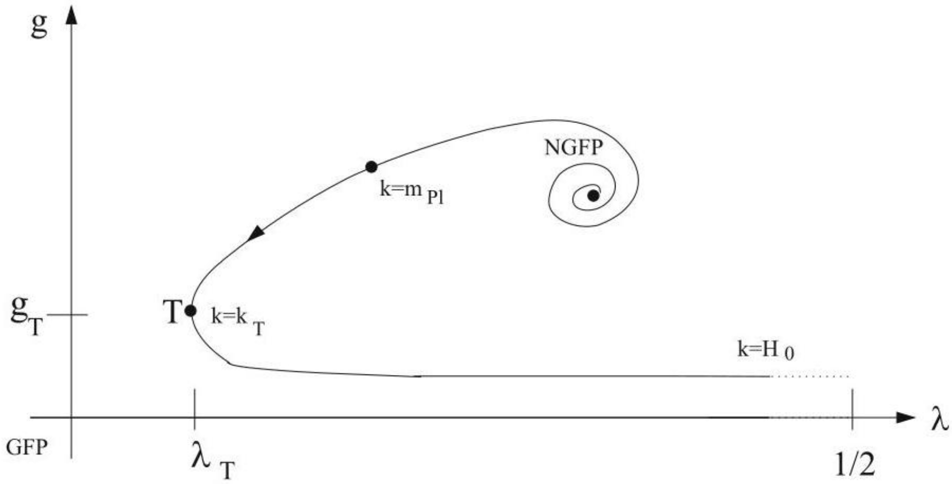


Fig. 1 A ”realistic” RG trajectory emanating from the NGFP and producing a classical era with a crossover near the GFP

图 1 一条从非高斯固定点 (NGFP) 出发，在高斯固定点 (GFP) 附近完成交叉、形成经典宇宙时期的“现实”重整化群轨迹

Up to this point we discussed only that segment of the ”trajectory realized in nature” which lies inside the linear regime of the GFP. The complete RG trajectory obtains by continuing this segment with the flow equation both into the IR and into the UV, where it ultimately spirals into the NGFP. While the UV continuation is possible within the Einstein-Hilbert truncation, this approximation breaks down in the IR when $\lambda(k)$ approaches 1/2 . Interestingly enough, this happens near $k = H_0$ , the present Hubble scale. The right panel of Fig. 1 shows a schematic sketch of the complete trajectory on the $g - \lambda$ -plane and Fig. 2 displays the resulting $t$ -dependence of $G$ and $\Lambda$ . Numerical integration of the field equations coupled with the $\beta$ -functions reveals a universe model which starts from a state of zero entropy and the radiation entropy observed today is due to the coarse graining. It turns out that the RG-improved field equations possess solutions with an epoch of power law inflation immediately after the initial singularity. The inflation is driven by the cosmological constant and ends automatically once the RG running has reduced the vacuum energy to the level of the matter energy density (see [23] for further details). Important extensions of these ideas to Bianchi IX and to the study of the BKL singularity have appeared in [24].

到目前为止，我们仅讨论了“自然界中实现的轨迹”处于 GFP 线性区域内的部分。通过流方程将该片段向红外和紫外方向延拓，即可得到完整的 RG 轨迹，轨迹最终会螺旋收敛于 NGFP。爱因斯坦-希尔伯特截断可以实现紫外方向的延拓，但当 $\lambda(k)$ 趋近 1/2 时，该近似在红外区域失效。有趣的是，这一失效发生在当前哈勃尺度 $k = H_0$ 附近。图 1 的右图给出了完整轨迹在 $g - \lambda$ 平面上的示意图，图 2 展示了由此得到的 $G$ 和 $\Lambda$ 对 $t$ 的依赖关系。对耦合 $\beta$ 函数的场方程进行数值积分，可得到一个起源于零熵状态的宇宙模型，如今观测到的辐射熵来源于粗粒化。结果表明，RG 改进的场方程存在初始奇点后随即经历幂律暴胀时期的解。该暴胀由宇宙学常数驱动，当 RG 跑动将真空能降低至物质能量密度的水平时，暴胀自动结束 (进一步细节见文献 [23])。这些思想在 Bianchi IX 型宇宙和 BKL 奇点研究中的重要推广见文献 [24]。

The approach discussed so far has the advantage of a direct physical interpretation of the averaging scale $1/k$ , but its actual implementation in a covariant framework presents some conceptual difficulties connected with the integrability condition of the Bianchi identities. For this reason an alternative approach based on a renormalization group improvement at the level of the action has been developed in [25-27] and applied in [28] to the problem of the $H_0$ tension [29]. Very recently, an alternative approach based only on the renormalization group improvement at the level of the equation of motion has recently appeared in [30], where bouncing solutions, recollapsing solutions, or non-singular expanding solutions with a transient acceleration era have been discussed in detail.

到目前为止讨论的这套方法优势在于可以直接对平均尺度 $1/k$ 给出物理解释，但在协变框架下实际实现时，会遇到与比安基恒等式可积性条件相关的一些概念性困难。因此，文献 [25-27] 发展了一种基于作用量层面重整化群改进的替代方法，并在文献 [28] 中被应用于 $H_0$ 张力问题 [29]。最近，文献 [30] 提出了另一种仅基于运动方程层面重整化群改进的方法，该文献详细讨论了弹跳解、再坍缩解，以及存在瞬态加速阶段的非奇异膨胀解。

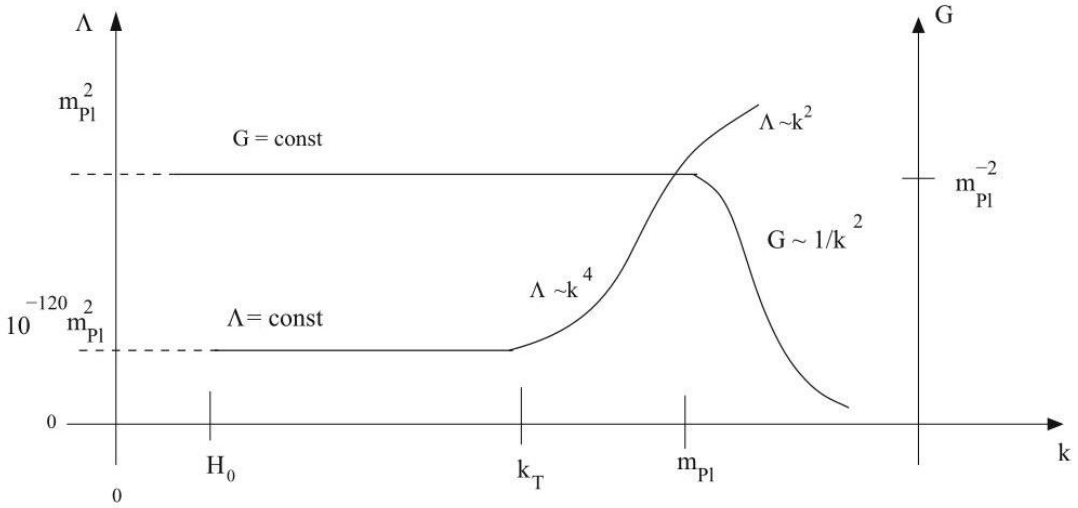


Fig. 2 Evolutionary trajectories of $G(t)$ and $\Lambda(t)$ emanating from the NGFP towards the infrared [23]

图 2 从 NGFP 向红外演化的 $G(t)$ 和 $\Lambda(t)$ 的演化轨迹 [23]

## Inflationary Models

## 暴胀模型

A direct connection between asymptotic safety and cosmological data may be best obtained in terms of an effective action for inflation as discussed by Weinberg [31], but here we make use of a different cutoff identification. As discussed before the Ricci curvature plays a role of a mass term in the fluctuation determinant of the FRGE; for this reason it is natural to assume that in the case of the early universe, $k$ is of the order of the local Ricci curvature, so that

渐近安全与宇宙学观测数据之间的直接联系，正如温伯格 [31] 所讨论的，或许可以通过暴胀的有效作用量最好地实现，但本文我们采用不同的截断标识。如前文所述，里奇曲率在 FRGE 的涨落行列式中起着质量项的作用；因此我们很自然地认为，在早期宇宙情形下，$k$ 约等于局部里奇曲率，因此

$$\Gamma_{k=k_{\text{inf}}}\left[g_{\mu\upsilon}\right],\ k_{\text{inf}}^2 \sim R. \tag{27}$$

In other words, fluctuations below $k^2 \sim R$ are suppressed and $R$ plays the role of an effective IR cutoff. If we assume that the scale of inflation is not too deep in the UV region, one can describe the flow in terms of its linear expansion around the NGFP, Eq. (13). In the presence of matter it is useful to distinguish between the two cases of real or complex conjugate critical exponents. In the first case (13) implies

换言之，能量低于 $k^2 \sim R$ 的涨落被压制，$R$ 充当了有效红外截断的角色。如果我们假设暴胀的能标不会深入紫外区域，就可以用它在 NGFP 附近的线性展开来描述流，即式 (13)。存在物质时，区分临界实指数与复共轭指数这两种情形是很有用的。第一种情形下，式 (13) 给出

$$\lambda_k = \lambda_* + c_1 e_1^1 \left(\frac{k}{k_0}\right)^{-\theta_1} + c_2 e_2^1 \left(\frac{k}{k_0}\right)^{-\theta_2} \tag{28}$$

$$g_k = g_* + c_1 e_1^2 \left(\frac{k}{k_0}\right)^{-\theta_1} + c_2 e_2^2 \left(\frac{k}{k_0}\right)^{-\theta_2} \tag{29}$$

For complex stability coefficients, $\theta_{1,2} = \theta' \pm i\theta''$ and the eigenvectors $\mathbf{e}_{1,2}$ are complex conjugates of each other. Redefining $\mathbf{e}_1 = \operatorname{Re}\mathbf{e}_1$ and $\mathbf{e}_2 = \operatorname{Im}\mathbf{e}_2$ and similarly for $c_{1,2}$ , the general solution can be written as

对于复稳定性系数，$\theta_{1,2} = \theta' \pm i\theta''$ 和特征向量 $\mathbf{e}_{1,2}$ 互为复共轭。重新定义 $\mathbf{e}_1 = \operatorname{Re}\mathbf{e}_1$ 和 $\mathbf{e}_2 = \operatorname{Im}\mathbf{e}_2$ ，对 $c_{1,2}$ 做类似处理后，通解可以写为

$$\lambda_k = \lambda_* + \Big((c_1 \cos(\theta'' t) + c_2 \sin(\theta'' t))\, e_1^1 \tag{30a}$$

$$+ (c_1 \sin(\theta'' t) - c_2 \cos(\theta'' t))\, e_2^1\Big) \left(\frac{k}{k_0}\right)^{-\theta'},$$

$$g_k = g_* + \Big((c_1 \cos(\theta'' t) + c_2 \sin(\theta'' t))\, e_1^2 \tag{30b}$$

$$+ (c_1 \sin(\theta'' t) - c_2 \cos(\theta'' t))\, e_2^2\Big) \left(\frac{k}{k_0}\right)^{-\theta'}.$$

Here $t \equiv \ln(k/k_0)$ . Therefore, assuming (27) we obtain two possible modifications to the Einstein-Hilbert Lagrangian. In the case of real critical exponents, the effective Lagrangian reads,

此处 $t \equiv \ln(k/k_0)$。因此，假设式 (27) 成立，我们可以得到爱因斯坦-希尔伯特拉格朗日量的两种可能修正。临界指数为实的情形下，有效拉格朗日量为:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{EH}} + a_0 R^2 + b_1 R^{\frac{4-\theta_1-\theta_2}{2}} + b_2 R^{\frac{4-\theta_1}{2}} + b_3 R^{\frac{4-\theta_2}{2}} + b_4 R^{2-\theta_1} + b_5 R^{2-\theta_2},$$

(31)

while in the case of complex critical exponents, one obtains

而临界指数为复的情形下，可以得到

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{EH}} + a_0 R^2 + \tilde{b}_1 R^{\frac{4-\theta'}{2}} \cos\left(\frac{1}{2}\theta'' \ln\left(\frac{\xi R}{k_0^2}\right)\right) + \tilde{b}_2 R^{\frac{4-\theta'}{2}} \sin\left(\frac{1}{2}\theta'' \ln\left(\frac{\xi R}{k_0^2}\right)\right)$$
$$+\tilde{b}_3 R^{2-\theta'} + \tilde{b}_4 R^{2-\theta'} \cos\left(\theta'' \ln\left(\frac{\xi R}{k_0^2}\right)\right) + \tilde{b}_5 R^{2-\theta'} \sin\left(\theta'' \ln\left(\frac{\xi R}{k_0^2}\right)\right)$$

(32)

where $a_0$ and the $b$’s and $\tilde{b}$’s are coefficients that must be determined from observations and $\mathcal{L}_{\text{EH}}$ is the standard Einstein-Hilbert action [32].

其中 $a_0$，$b$ 和 $\tilde{b}$ 是必须由观测确定的系数，$\mathcal{L}_{\text{EH}}$ 是标准的爱因斯坦-希尔伯特作用量 [32]。

Can we test the consistency of the scale setting procedure we have employed? This is in general a very difficult task because the solution of the flow equation (6) for the $f(R)$ theory at $k = 0$ is not available. On the other hand, the properties of the fixed-point solution at a large curvature have been studied. As discussed in [33] for the Benedetti-Caravelli flow equation [34], near the NGFP the structure of the fixed-point scaling Lagrangian reads

我们能否检验所用标度设定流程的自洽性？一般而言这是一项非常困难的任务，因为 $f(R)$ 理论在 $k = 0$ 处的流方程 (6) 没有通解。另一方面，大曲率下不动点解的性质已有研究。正如文献 [33] 针对贝内代蒂-卡拉韦利流方程 [34] 的讨论，在 NGFP 附近，不动点标度拉格朗日量的结构为

$$\mathcal{L}_{\text{eff}} = aR^2 + R\left(\frac{3a}{2} + b\cos\ln R^2 + c\sin\ln R^2\right) \tag{33}$$

which can be considered a special case of (32) depending on the value of the critical exponents and of the constants $a, b$ and $c$. One can thus hope that, at least qualitatively, the scale setting procedure based on the $k^2 \propto R$ identification at the level of the action leads to physically meaningful results. Following this idea we can consider the following Lagrangian which includes the relevant coupling $R^2$,

它可以看作式 (32) 的特殊情形，依赖于临界指数以及常数 $a, b$ 和 $c$ 的取值。因此我们可以预期，至少在定性层面，基于作用量层面 $k^2 \propto R$ 标识的标度设定流程能得到具有物理意义的结果。遵循这一思路，我们可以考虑如下包含相关耦合 $R^2$ 的拉格朗日量:

$$\mathcal{L}_k = \frac{1}{16\pi G_k}(R - 2\Lambda_k) - \beta_k R^2 \tag{34}$$

and implement the cutoff identification in (27). A detailed calculation shows that the resulting effective action can be obtained [35]:

并实现式 (27) 中的截断标识。详细计算表明，可以得到最终的有效作用量 [35]:

$$S=\frac{1}{2\pi\kappa^2}\int d^4x\sqrt{-g}\left[R+\alpha R^{2-\frac{\theta_3}{2}}+\frac{R^2}{6m^2}-\Lambda\right]. \tag{35}$$

Here $m$ is the scalaron mass, $\theta_3$ is the critical exponent of the $R^2$ -operator, and $\kappa^2 = 8\pi G$ . As $\theta_3 \approx 1$ , it is then possible to constrain the value of $\alpha$ in the slow-roll approximation. Mapping (35) to the Einstein frame yields

此处 $m$ 是标量子质量，$\theta_3$ 是 $R^2$ 算符的临界指数，且 $\kappa^2 = 8\pi G$ 。在慢滚近似下，结合 $\theta_3 \approx 1$ ，就可以对 $\alpha$ 的取值给出限制。将式 (35) 映射到爱因斯坦架得到

$$S=\int d^4x\sqrt{-g_E}\left[\frac{1}{2\kappa^2}R_E-\frac{1}{2}g_E^{\mu\upsilon}\partial_\mu\phi\partial_\upsilon\phi-V_\pm(\phi)\right] \tag{36}$$

where

其中

$$V_\pm(\phi)=\frac{m^2\mathrm{e}^{-2\sqrt{\frac{2}{3}}\kappa\phi}}{256\kappa^2}\Bigg\{192\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\kappa\phi}-1\right)^2$$

$$-3\alpha^4+128\Lambda-3\alpha^2\left(\alpha^2+16\mathrm{e}^{\sqrt{\frac{2}{3}}\kappa\phi}-16\right)m_{\mathrm{Pl}}6\alpha^3\sqrt{\alpha^2+16\mathrm{e}^{\sqrt{\frac{2}{3}}\kappa\phi}-16}$$

$$-\sqrt{32}\alpha\left[\left(\alpha^2+8\mathrm{e}^{\sqrt{\frac{2}{3}}\kappa\phi}-8\right)\pm\alpha\sqrt{\alpha^2+16\mathrm{e}^{\sqrt{\frac{2}{3}}\kappa\phi}-16}\right]^{\frac{3}{2}}\Bigg\}.$$

(37)

Here $\alpha$ and $\Lambda$ are dimensionless effective coupling constants, obtained by rescaling $\alpha \to \alpha/3\sqrt{3}m$ and $\Lambda \to \Lambda m^2$ . While in the limit $\alpha \to 0$ one recovers the original Starobinsky model, in general a nonzero $\alpha$ introduces a significant deformation of that model where two different types of scalaron potentials are possible.

此处 $\alpha$ 和 $\Lambda$ 是无量纲有效耦合常数，由 $\alpha \to \alpha/3\sqrt{3}m$ 和 $\Lambda \to \Lambda m^2$ 标度变换得到。当 $\alpha \to 0$ 取极限时，我们可以回到原始斯塔罗宾斯基模型，而一般情况下，非零的 $\alpha$ 会对该模型产生显著的形变，产生两种不同类型的标量子势。

As discussed in [35] it turns out that it is possible to constrain the value of $\alpha$ in the slow-roll approximation and in particular for $\alpha \in [1,3]$ and $N = 50$ e-folds, the spectral index $n_s \in (0.965, 0.967)$ and the tensor-to-scalar ratio $r \in (0.069, 0.0076)$. These values are significantly larger than the Starobinsky value but still in agreement with observations [36]. Future CMB anisotropy experiments like CORE [37] or LiteBIRD [38] should be able to discriminate among these models.

正如文献 [35] 所讨论，在慢滚近似下，特别是对 $\alpha \in [1,3]$ 和 $N = 50$ e 圈折叠数而言，我们可以对 $\alpha$ 的值加以约束，得到谱指数 $n_s \in (0.965, 0.967)$ 与张标比 $r \in$ 分别为 (0.069, 0.0076)。这些值明显大于斯塔罗宾斯基模型的预测值，但仍与观测结果 [36] 相符。未来的 CMB 各向异性实验，如 CORE[37] 或 LiteBIRD[38]，应当能够区分这些模型。

## Markov-Mukhanov Approach and Bouncing Cosmologies

## 马尔可夫-穆哈诺夫方法与反弹宇宙学

In this section we shall make use of the Markov-Mukhanov formalism [39] to discuss a class of bouncing cosmologies with a dynamically evolving Newton's constant from the AS scenario.

本节我们将利用马尔可夫-穆哈诺夫形式体系 [39]，讨论一类来自 AS 方案、牛顿常数随动力学演化的反弹宇宙学。

Let us consider a matter fluid of energy density $\varepsilon, 4$ -velocity $u^\mu$ with $u_\mu u^\mu=-1$ and rest-mass density $\rho$ . Mass continuity implies

我们考察能量密度为 $\varepsilon, 4$ 、速度为 $u^\mu$ 的物质流体，满足 $u_\mu u^\mu=-1$ ，静止质量密度为 $\rho$ 。质量连续性给出

$$(\rho u^\mu);\mu=0 \tag{38}$$

and

和

$$\frac{\delta\rho}{\rho}=\frac{\delta\varepsilon}{p(\varepsilon)+\varepsilon} \tag{39}$$

for a non-dissipative fluid (note that in general $\varepsilon=\varepsilon(\rho)$ ). In the presence of gravity the field equations can be derived from the following action:

适用于无耗散流体 (注意一般情况下 $\varepsilon=\varepsilon(\rho)$ )。存在引力时，场方程可由如下作用量导出:

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[R+2\chi(\varepsilon)\mathcal{L}] \tag{40}$$

where $\mathcal{L}=-\varepsilon$ is the matter Lagrangian and the function $\chi=\chi(\varepsilon)$ is a multiplicative gravity-matter coupling. The metric variation of the matter part of the Lagrangian yields

其中 $\mathcal{L}=-\varepsilon$ 是物质拉格朗日量，函数 $\chi=\chi(\varepsilon)$ 是引力-物质乘法耦合。拉格朗日量物质部分对度规的变分给出

$$\frac{1}{\sqrt{-g}}\delta\left(2\sqrt{-g}\chi\varepsilon\right)=2(\chi\varepsilon)'\delta\varepsilon-\chi\varepsilon g_{\mu\nu}\delta g^{\mu\nu} \tag{41}$$

and prime means derivative with respect to $\varepsilon$ . The variation of $\rho$ under a change of the metric reads [40]

撇表示对 $\varepsilon$ 求导。$\rho$ 对度规变化的变分由文献 [40] 给出

$$\delta\rho=\frac{\rho}{2}\left(g_{\mu\nu}+u_\mu u_\nu\right)\delta g^{\mu\nu} \tag{42}$$

so that the total variation of the action (40) leads to the following field equations:

因此对作用量 (40) 做总变分后得到如下场方程:

$$R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R = (\chi\varepsilon)' T_{\mu\upsilon} + \chi'\varepsilon^2 g_{\mu\upsilon}. \tag{43}$$

Here we must assume that $\chi(\varepsilon = 0) = 8\pi G_N$ to recover standard GR at low energies, $T_{\mu\nu}$ is the energy-momentum tensor for a perfect fluid

此处我们必须假设 $\chi(\varepsilon = 0) = 8\pi G_N$ ，才能在低能下恢复标准广义相对论，$T_{\mu\nu}$ 是理想流体的能量动量张量

$$T_{\mu\upsilon} = (\varepsilon + p(\varepsilon)) u_\mu u_\upsilon + \varepsilon g_{\mu\upsilon}, \tag{44}$$

and we used (39). Bianchi identities imply

并且我们使用了 (39) 式。比安基恒等式给出

$$(\chi\varepsilon)'' \varepsilon_{,\mu} T^{\mu\nu} + \left(\chi'\varepsilon^2\right) \varepsilon^{,\nu} = 0 \tag{45}$$

as we used $T_{\mu\nu}{}^{;\mu} = 0$ . By projecting along $u^\mu$ we obtain an identity, and therefore, the total effective energy-momentum tensor

推导中用到了 $T_{\mu\nu}{}^{;\mu} = 0$ 。沿 $u^\mu$ 投影后我们得到一个恒等式，因此总有效能量动量张量

$$T_{\mu\nu}^{\text{eff}} = (\chi\varepsilon)' T_{\mu\nu} + \chi'\varepsilon^2 g_{\mu\nu} \tag{46}$$

is always conserved in a homogeneous universe. We can thus specialize the action for a homogeneous FRW background so that

在均匀宇宙中总是守恒的。我们因此可以将作用量特殊化为均匀 FRW 背景，得到

$$S =: \frac{3V_3}{\kappa^2} \int dt L = \frac{3V_3}{\kappa^2} \int dt \left( a\dot{a} - aK + \frac{a^3}{3} \int_0^\varepsilon G(s)\, ds \right) \tag{47}$$

where $V_3$ is the volume of the 3-space, $K$ the intrinsic curvature parameter, and $L = L(a, \dot{a})$ the Lagrangian. If $\pi_a = \delta L/\delta\dot{a}$ is the conjugate momentum to $a(t)$, the condition on the Hamiltonian $H = \pi_a\dot{a} - L = 0$ implies that

其中 $V_3$ 是三维空间的体积，$K$ 是内禀曲率参数，$L = L(a, \dot{a})$ 是拉格朗日量。若 $\pi_a = \delta L/\delta\dot{a}$ 是 $a(t)$ 的共轭动量，哈密顿量 $H = \pi_a\dot{a} - L = 0$ 满足的条件给出

$$\dot{a}^2 + K + V(a) = 0 \tag{48}$$

where

其中

$$V(a) = -\frac{a^2}{3}\int_0^{\varepsilon(a)} G(s)\, ds \tag{49}$$

and $\varepsilon(a)$ is determined by the conservation law

且 $\varepsilon(a)$ 由守恒定律确定

$$d\varepsilon + 3\left(p(\varepsilon) + \varepsilon\right) d\ln a = 0. \tag{50}$$

We now need to specify both an EOS for $p(\varepsilon)$ and $G(\varepsilon)$ . If we assume that the energy scale is determined by the energy density of the system, the NGFP is reached at infinite densities, so that in general $G = G(\varepsilon/\varepsilon_0)$ and $\lim_{\varepsilon/\varepsilon_0 \to \infty} G(\varepsilon/\varepsilon_0) = 0$ being $\varepsilon_0$ the Plank energy density. Moreover, $\lim_{\varepsilon/\varepsilon_0 \to 0} G(\varepsilon/\varepsilon_0) = G_{\mathrm{N}}$ where $G_{\mathrm{N}}$ is the value of the Newton constant. We can encode this behavior in a functional form of the type

现在我们需要同时指定 $p(\varepsilon)$ 和 $G(\varepsilon)$ 的物态方程。如果假设系统的能标由自身能量密度决定，那么非高斯不动点 (NGFP) 在无穷密度处达到，因此一般有 $G = G(\varepsilon/\varepsilon_0)$ ，其中 $\varepsilon_0$ 是普朗克能量密度，$\lim_{\varepsilon/\varepsilon_0 \to \infty} G(\varepsilon/\varepsilon_0) = 0$ 成立。此外，$\lim_{\varepsilon/\varepsilon_0 \to 0} G(\varepsilon/\varepsilon_0) = G_{\mathrm{N}}$ ，其中 $G_{\mathrm{N}}$ 是牛顿常数的值。我们可以将这一行为表示为如下形式:

$$G(\varepsilon/\varepsilon_0) = \frac{G_{\mathrm{N}}}{1 + \xi\left(\frac{\varepsilon}{\varepsilon_0}\right)^{\gamma}} \tag{51}$$

which loosely resembles the result of the $\beta$ -functions in some models of AS [2], and $\xi$ and $\gamma$ are positive free parameters.

这与渐近安全 (AS) 部分模型中 $\beta$ 函数的结果大致相似，且 $\xi$ 和 $\gamma$ 为正自由参数。

For an equation of state of the type $p = w\varepsilon$ , bouncing or emerging cosmologies are then obtained for various values of $\gamma$ and $w$ , as in Figs. 3 and 4 for the potential $V(a)$ . It would be interesting to discuss the phenomenological implication of this cosmological model in terms of structure formation and initial spectrum of cosmological perturbations.

对于物态方程形式为 $p = w\varepsilon$ 的情况，当 $\gamma$ 和 $w$ 取不同值时即可得到反弹宇宙学或涌现宇宙学，如势场 $V(a)$ 对应的图 3 和图 4 所示。讨论该宇宙学模型对结构形成和宇宙扰动原初谱的唯象启示会是很有意义的工作。

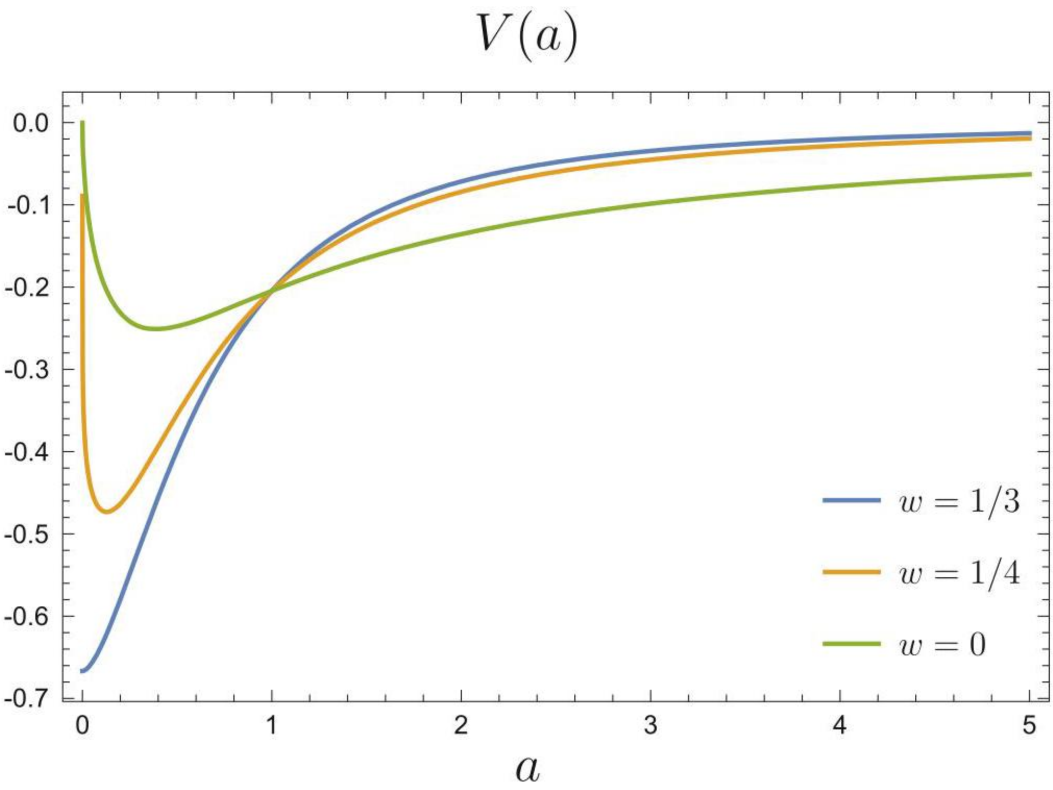


Fig. 3 Effective potential for $K = 1, \gamma = 1/2$ , and various values of $w$ . Note that in this case there is no bounce for $w > 1/3$ but an emergent universe

图 3 $K = 1, \gamma = 1/2$ 对应的有效势，以及 $w$ 取不同值的情况。注意在该情况下，当参数为 $w > 1/3$ 时不存在反弹，得到的是一个涌现宇宙。

## The IR Fixed-Point Cosmology

## 红外不动点宇宙学

In section ”Wilsonian Action in Cosmology”, the presence of a singularity in the $\beta$ -functions (10) at a finite value of $k = k_{\text{term}}$ when $\lambda(k_{\text{term}}) = 1/2$ does not allow to reach the $k \to 0$ physical limit. The termination of the flow at finite value of $k$ could be simply a consequence of the nonuniversal feature of the flow in the IR, in fact if the pole at $\lambda = 1/2$ disappears for a class of spectrally adjusted type III cutoff [41] or for the exponential parametrization of the fluctuation field [42]. On the other hand, the presence of the pole could also signal the occurrence of a phase transition as $k \to 0$ where a new set of IR-relevant operator emerges in the effective action, as it is well known, for instance, in a scalar field theory in the broken phase. In order to illustrate this point, let us consider a $\mathcal{Z}_2$ -symmetric real scalar field,

在“宇宙学中的威尔逊作用量”一节中，当 $\lambda(k_{\text{term}}) = 1/2$ 无法达到 $k \to 0$ 物理极限时，$\beta$ β 函数在 $k = k_{\text{term}}$ 的有限值处存在奇点。流在有限 $k$ 值处终止，可能只是红外流非普适特性的结果；事实上，对于一类经谱调整的 III 型截断 [41]，或是对涨落场采用指数参数化 [42]，$\lambda = 1/2$ 处的极点就会消失。另一方面，极点的存在也可能预示着相变发生，进入 $k \to 0$ 后有效作用量中会涌现一组新的红外相关算符——这一点广为人知，例如破缺相中的标量场论就是如此。为说明这一点，我们考虑一个 $\mathcal{Z}_2$ 对称的实标量场，

$$S_k[\phi] = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2(k)\phi^2 + \frac{1}{12}\lambda(k)\phi^4 \right\}. \tag{52}$$

In a momentum representation we have

在动量表象下我们有

$$S_k^{(2)} = \frac{\delta^2 S_k}{\delta\phi^2} = p^2 + m^2(k) + \lambda(k)\phi^2, \tag{53}$$

so that $S_k^{(2)}$ is positive if $m^2 > 0$ , but when $m^2 < 0$ , it can become negative for $\phi^2$ small enough. Of course, the negative eigenvalue for $\phi = 0$ , for example, indicates that the fluctuations are unstable, and the nonlinear evolution of this instability is a ”condensation” which shifts the field from the ”false vacuum” to the true one, a phenomenon called instability-induced renormalization [43]. In particular, the $\boldsymbol{\beta}$ -functions, obtained by $p$ -integrals over (powers of) the propagator $\left[p^2 + m^2(k) + k^2\right]^{-1}$ are regular in the symmetric phase $\left(m^2 > 0\right)$ , but there is a pole at $p^2 = -m(k)^2 - k^2$ provided $k^2$ is small enough in the broken phase $\left(m^2 < 0\right)$. For $k^2 \searrow \left|m(k)^2\right|$ , the $\boldsymbol{\beta}$ -functions become large and there the instability-induced renormalization occurs. In a reliable truncation, a physically realistic RG trajectory in the spontaneously broken regime will not hit the singularity at $k^2 = \left|m(k)^2\right|$ , but rather make $m(k)$ run in such a way that $\left|m(k)^2\right|$ is always smaller than $k^2$ . This requires that

因此当 $m^2 > 0$ 时 $S_k^{(2)}$ 为正，但当 $m^2 < 0$ 时，只要 $\phi^2$ 足够小，$S_k^{(2)}$ 就可以变为负。显然，$\phi = 0$ 的负本征值意味着涨落不稳定，这种不稳定性的非线性演化会引发“凝聚”，使场从“假真空”移动到真真空，这一现象称为不稳定性诱导重整化 [43]。具体来说，通过对传播子 $\left[p^2 + m^2(k) + k^2\right]^{-1}$ (的幂次) 做 $p$ 积分得到的 $\boldsymbol{\beta}$ β 函数在对称相 $\left(m^2 > 0\right)$ 中是正则的，但在破缺相 $\left(m^2 < 0\right)$ 中，只要 $k^2$ 足够小，就会在 $p^2 = -m(k)^2 - k^2$ 处出现极点。当 $k^2 \searrow \left|m(k)^2\right|$ 时，$\boldsymbol{\beta}$ β 函数会变大，不稳定性诱导重整化在此发生。在可靠的截断下，自发破缺区域中符合物理实际的 RG 轨迹不会撞上 $k^2 = \left|m(k)^2\right|$ 处的奇点，而是会让 $m(k)$ 以 $\left|m(k)^2\right|$ 始终小于 $k^2$ 的方式跑动。这要求

$$-m(k)^2 \propto k^2 . \tag{54}$$

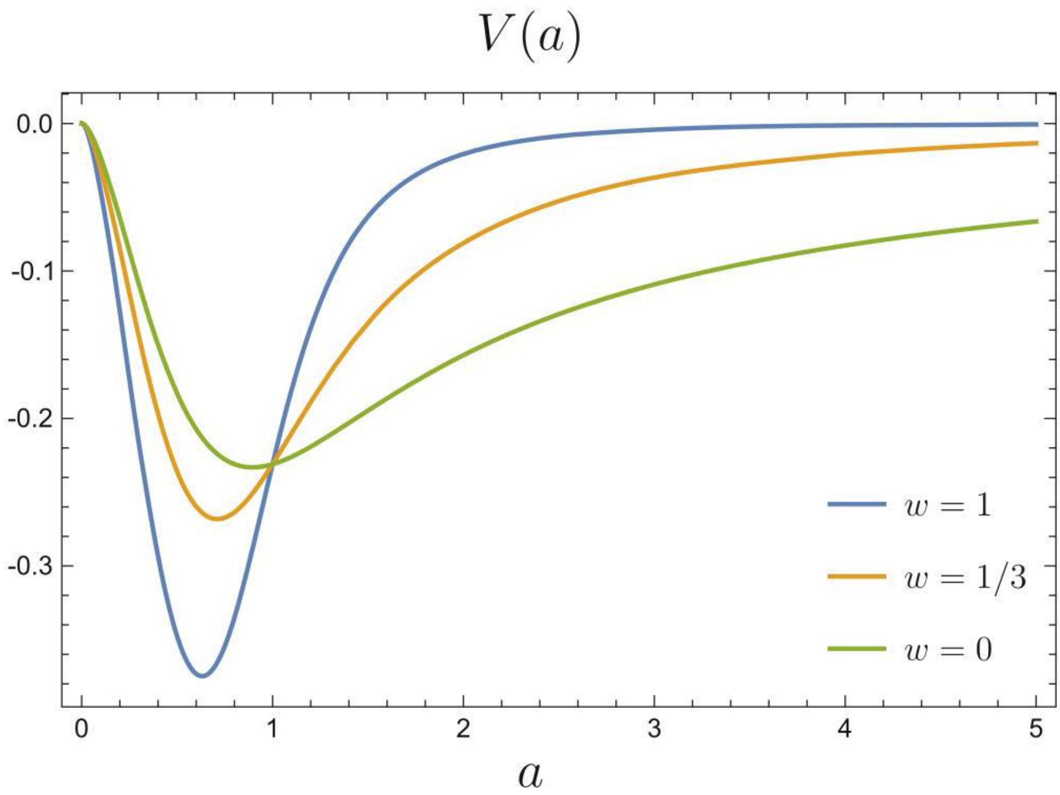


Fig. 4 Same as the previous figure, but in this case $\gamma = 1$ . In this case a bounce is always present

图 4 与前一图相同，但本图中 $\gamma = 1$ ，这种情况下始终存在回弹

In order to avoid the singularity, a special running of the mass term is necessary to properly recover a convex effective potential in the $k \to 0$ limit [26]. However, the truncation implied in (52) is not enough to describe the broken phase, because its RG trajectories terminate at a finite scale $k_{\text{term}}$ with $k^2_{\text{term}} = \left|m(k_{\text{term}})^2\right|$ at which the $\boldsymbol{\beta}$ -functions diverge. Instead, if one allows for an arbitrary running potential $U_k(\phi)$ , containing infinitely many couplings, all trajectories can be continued to $k = 0$ , and for $k \to 0$ one finds indeed the quadratic mass renormalization (54).

为了避免奇点，质量项需要特殊的跑动，才能在 $k \to 0$ 极限下正确得到凸有效势 [26]。但 (52) 隐含的截断不足以描述破缺相，因为它的 RG 轨迹会在有限标度 $k_{\text{term}}$ 终止于 $k^2_{\text{term}} = \left|m(k_{\text{term}})^2\right|$ ，此时 $\boldsymbol{\beta}$ β 函数发散。反之，如果允许任意跑动势 $U_k(\phi)$ 包含无穷多个耦合，所有轨迹都可以延拓到 $k = 0$ ，并且对于 $k \to 0$ 确实可以得到二次质量重整化 (54)。

How is this example related to gravity? Let us consider a family of "off-shell," spherically symmetric backgrounds labeled by the radius of the sphere $\phi$ , in order to disentangle the contributions from the two invariants $\int d^4x\sqrt{g} \propto \phi^4$ and $\int d^4x\sqrt{g}R \propto \phi^2$ to the Einstein-Hilbert flow. It is then convenient to decompose the fluctuation $h_{\mu\nu}$ on the sphere into irreducible components and to expand the irreducible pieces in terms of the corresponding spherical harmonics. For $h_{\mu\nu}$ in the transverse-traceless (TT) sector, the operator $\Gamma_k^{(2)}$ equals, up to a positive constant,

这个例子和引力有何关联？为了区分两个不变量 $\int d^4x\sqrt{g} \propto \phi^4$ 和 $\int d^4x\sqrt{g}R \propto \phi^2$ 对爱因斯坦-希尔伯特流的贡献，我们考虑一族由球半径 $\phi$ 标记的“离壳”球对称背景。将球面上的涨落 $h_{\mu\upsilon}$ 分解为不可约分量，再将不可约部分按相应球谐函数展开，这样处理是方便的。对于横向无迹 (TT) 区的 $h_{\mu\upsilon}$，算符 $\Gamma_k^{(2)}$ 在一个正常数范围内等于

$$-\nabla^2 + 8\phi^{-2} + k^2 - 2\Lambda(k), \tag{55}$$

where $\nabla^2 \equiv g^{\mu\upsilon}\nabla_\mu\nabla_\upsilon$ is the covariant Laplacian acting on TT tensors. The spectrum of $-\nabla^2$, denoted $\{p^2\}$, is discrete and positive. Clearly,(55) is a positive operator if the cosmological constant is negative. In this case there are only stable, bounded oscillations, leading to a mild fluctuation-induced renormalization. The situation is very different for $\Lambda > 0$ where, for $k^2$ sufficiently small,(55) has negative eigenvalues, i. e., unstable eigenmodes. The allowed part of the $g$ - $\lambda$ -plane ($\lambda < 1/2$) corresponds to the situation $k^2 > 2\Lambda(k)$ where the singularity is avoided thanks to the large regulator mass. When $k^2$ approaches $2\Lambda(k)$ from above the $\boldsymbol{\beta}$ -functions become large and strong renormalizations set in, driven by the modes that would go unstable at $k^2 = 2\Lambda$. In this respect the situation is completely analogous to the scalar theory discussed above: its symmetric phase $\left(m^2 > 0\right)$ corresponds to gravity with $\Lambda < 0$; in this case all fluctuation modes are stable and only small renormalization effects occur. Conversely, in the broken phase $\left(m^2 < 0\right)$ and in gravity with $\Lambda > 0$, there are modes, which are unstable in the absence of the IR regulator. They lead to strong IR renormalization effects for $k^2 \searrow \left|m(k)^2\right|$ and $k^2 \searrow 2\Lambda(k)$, respectively. It is therefore possible to conclude that the instability-induced renormalization should occur also in this framework so that

其中 $\nabla^2 \equiv g^{\mu\upsilon}\nabla_\mu\nabla_\upsilon$ 是作用于 TT 张量的协变拉普拉斯算符。$-\nabla^2$ 的谱记为 $\{p^2\}$，是离散正定的。显然，若宇宙常数为负，(55) 是一个正算符。这种情况下只存在稳定的有界振荡，仅产生温和的涨落诱导重整化。当处于 $\Lambda > 0$ 的情况时，情况截然不同: 对于足够小的 $k^2$，(55) 存在负特征值，即不稳定本征模。$g$ - $\lambda$ 平面 ($\lambda < 1/2$) 的允许区域对应 $k^2 > 2\Lambda(k)$ 这种情况: 大调节质量帮助我们避免了奇点。当 $k^2$ 从上方趋近 $2\Lambda(k)$ 时，$\boldsymbol{\beta}$ 函数变大，强重整化开始出现，这是由本在 $k^2 = 2\Lambda$ 处就会变得不稳定的模式驱动的。就此而言，这里的情况和我们上文讨论的标量理论完全类似: 它的对称相 $\left(m^2 > 0\right)$ 对应具有 $\Lambda < 0$ 的引力；此时所有涨落模式都是稳定的，仅产生很小的重整化效应。反之，在破缺相 $\left(m^2 < 0\right)$ 和具有 $\Lambda > 0$ 的引力中，若不存在红外调节器，就会存在不稳定模式。它们分别会给 $k^2 \searrow \left|m(k)^2\right|$ 和 $k^2 \searrow 2\Lambda(k)$ 带来强红外重整化效应。因此我们可以得出结论: 这种不稳定性诱导的重整化也应该会在该框架中出现，即

$$\lim_{k\to 0}\lambda(k) = \lambda_*^{\mathrm{IR}},\ \lim_{k\to 0} g(k) = g_*^{\mathrm{IR}} \tag{56}$$

with $\lambda_*^{\mathrm{IR}} \neq 0$ and $g_*^{\mathrm{IR}} \neq 0$. Investigations based on a bimetric truncation of Einstein-Hilbert gravity have actually found a fixed point which could regulate IR behavior [44]. Moreover, there are also indications [45] that quantum Einstein gravity, because of its inherent IR divergences, is subject to strong renormalization effects also at very large distances. In cosmology those effects would be relevant to the universe at late times. It has been speculated that they might lead to a dynamical relaxation of $\Lambda$, thus solving the cosmological constant problem [45,46]. An analysis of such IR effects in the framework of the effective average action is not available yet. It would require truncations which are much more complicated than the standard Einstein-Hilbert action, which contain nonlocal invariants, for instance.

其中包含 $\lambda_*^{\mathrm{IR}} \neq 0$ 和 $g_*^{\mathrm{IR}} \neq 0$ 。实际上，基于爱因斯坦-希尔伯特引力双度量截断的研究已经发现了一个可以调节红外行为的不动点 [44]。此外，也有研究表明 [45]，量子爱因斯坦引力因其固有的红外发散，即使在极远的距离也会受到强重整化效应的影响。在宇宙学中，这些效应对晚期宇宙至关重要。有猜测认为它们可能会导致 $\Lambda$ 的动力学弛豫，从而解决宇宙常数问题 [45,46]。目前还没有在有效平均作用量框架下对这类红外效应的分析。这类分析需要比标准爱因斯坦-希尔伯特作用量复杂得多的截断，例如需要包含非局域不变量。

The postulated fixed point is the IR counterpart of the UV attractive non-Gaussian fixed point which is known to exist in the Einstein-Hilbert truncation of pure quantum gravity. More generally, we assume that the exact cosmologically relevant RG trajectory in $(g, \lambda)$ -space smoothly interpolates between $\left(g_*^{\mathrm{UV}}, \lambda_*^{\mathrm{UV}}\right)$ for $k \rightarrow \infty$ and $\left(g_*^{\mathrm{IR}}, \lambda_*^{\mathrm{IR}}\right)$ for $k \rightarrow 0$ . The UV fixed point is important for the very early universe $(t \rightarrow 0)$ , while the IR fixed point determines the cosmology at late times $(t \rightarrow \infty)$ . It is important to stress that a similar crossover between two nontrivial RG fixed points has actually been shown to exist in two-dimensional Liouville quantum gravity [47]. Its RG trajectory connects two conformal field theories with central charges $25-c$ and $26-c$ , respectively, where $c$ is the central charge of the matter system.

这一假设的不动点是紫外吸引非高斯不动点的红外对应,后者已被证实在纯量子引力的爱因斯坦-希尔伯特截断中存在。更普遍地说，我们假设 $(g, \lambda)$ 空间中确切的宇宙学相关重整化群轨迹在 $k \rightarrow \infty$ 对应的 $\left(g_*^{\mathrm{UV}}, \lambda_*^{\mathrm{UV}}\right)$ 与 $k \rightarrow 0$ 对应的 $\left(g_*^{\mathrm{IR}}, \lambda_*^{\mathrm{IR}}\right)$ 之间平滑过渡。紫外不动点对极早期宇宙至关重要 $(t \rightarrow 0)$ ，而红外不动点决定了晚期宇宙的演化状态 $(t \rightarrow \infty)$ 。需要强调的是，二维刘维尔量子引力中实际上已经证实存在类似的两个非平庸重整化群不动点之间的交叉 [47]。其重整化群轨迹分别连接中心荷为 $25-c$ 和 $26-c$ 的两个共形场论，其中 $c$ 是物质系统的中心荷。

## Late Universe Dynamics from the IRFP Scaling

## 来自 IRFP 标度的晚期宇宙动力学

It is interesting to see the consequences of the IRFP for the late time dynamics of the universe. We can assume in Eq. (22) that the entropy production is zero far from the NGFP, and therefore, homogeneous and isotropic cosmologies can be described by the following system of equations:

探究红外不动点 (IRFP) 对宇宙晚期动力学的影响十分有意义。我们可以在式 (22) 中假设，远离非高斯不动点 (NGFP) 处熵产生为零，因此均匀各向同性宇宙可以由下述方程组描述:

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}=\frac{1}{3} \Lambda+\frac{8 \pi}{3} G \rho \tag{57}$$

$$\dot{\rho}+3(1+w)\left(\frac{\dot{a}}{a}\right) \rho=0 \tag{58}$$

$$\dot{\Lambda}+8 \pi \rho \dot{G}=0 \tag{59}$$

$$G(t) \equiv G(k=k(t)),\ \Lambda(t) \equiv \Lambda(k=k(t)) . \tag{60}$$

Equation (57) is the standard Friedmann equation with a time-dependent $\Lambda$ and $G$ , and Eq. (58) expresses the conservation of $T^{\mu\upsilon}$ . Equation (59) is a consistency condition which is dictated by Bianchi's identity. It guarantees that the RHS of Einstein's equation has vanishing covariant divergence.

式 (57) 是含时 $\Lambda$ 和 $G$ 的标准弗里德曼方程，式 (58) 表述了 $T^{\mu\upsilon}$ 的守恒律。式 (59) 是由比安基恒性给出的自洽条件，它保证爱因斯坦方程右手项的协变散度为零。

Let us assume that we know the functions $G(k)$ and $\Lambda(k)$ for $k \to 0$ , i.e., in the IR. The idea is to express the mass parameter $k$ in terms of the physically relevant cutoff scale. We can assume

我们假设已知红外区即 $k \to 0$ 对应的 $G(k)$ 和 $\Lambda(k)$ 函数。我们的思路是将质量参数 $k$ 用物理相关的截断标度表示。我们可以假设

$$k(t) = \xi/t \tag{61}$$

where $\xi > 0$ is an a priori unknown constant. Note that for power-law solutions, the scale factor identification in (61) is equivalent to (24). Inserting (61) into $G(k)$ and $\Lambda(k)$ , we obtain the time-dependent quantities $G(t) \equiv G(k = \xi/t)$ and $\Lambda(t) \equiv \Lambda(k = \xi/t)$ where the cutoff identification (61) applies only in the case of perfect homogeneity and isotropy for which $k_{\text{cosmo}} \equiv k(t) = \xi/t$ is the only relevant scale. Allowing for (large, nonlinear) density perturbations $\delta\rho(\mathbf{x}, t)$ of a typical wavelength $\lambda_{\text{pert}}$ , we introduce a new scale $k_{\text{pert}} = 2\pi/\lambda_{\text{pert}}$ into the problem. Similarly, immersing a localized matter distribution (a massive body) of total mass $M$ into the cosmological fluid gives rise to the scale $k_{\text{M}} = M$ . In the situations of interest, the latter two mass scales are much larger than the cosmological one: $k_{\text{pert}}, k_{\text{M}} \gg k_{\text{cosmo}}$ .

其中 $\xi > 0$ 是一个先验未知常数。注意，对于幂律解，标度因子的定义式 (61) 与式 (24) 等价。将 (61) 代入 $G(k)$ 和 $\Lambda(k)$ , 我们得到含时物理量 $G(t) \equiv G(k = \xi/t)$ 和 $\Lambda(t) \equiv \Lambda(k = \xi/t)$ ; 其中截断定义式 (61) 仅适用于完全均匀各向同性的情况，此时 $k_{\text{cosmo}} \equiv k(t) = \xi/t$ 是唯一相关标度。若存在特征波长为 $\lambda_{\text{pert}}$ 的 (大尺度非线性) 密度扰动 $\delta\rho(\mathbf{x}, t)$ ，我们就给问题引入了一个新标度 $k_{\text{pert}} = 2\pi/\lambda_{\text{pert}}$ 。类似地，将总质量为 $M$ 的局域物质分布 (一个大质量天体) 嵌入宇宙流体，会产生标度 $k_{\text{M}} = M$ 。在我们关心的情景中，后两种质量标度远大于宇宙学质量标度: $k_{\text{pert}}, k_{\text{M}} \gg k_{\text{cosmo}}$ 。

The assumption for the RG flow of $G$ and $\Lambda$ at "non-cosmological" scales $k \gg k_{\text{cosmo}}$ (e.g., for $k \approx k_{\text{pert}}, k_M$ ) is that at those scales the $k$ -dependence is very weak or zero so that standard gravity is recovered at sub-cosmological scales. Therefore, for $k \lesssim k_{\text{cosmo}}$ we assume the validity of the IR fixed-point behavior (56). For $k \gtrsim k_{\text{cosmo}}$ the assumption is that $G$ and $\Lambda$ depend on $k$ only extremely weakly or are $k$ -independent. In this manner we recover standard gravity with $G, \Lambda = \text{const}$ at length scales smaller than the cosmological scale $\propto t$ . In particular $G$ and $\Lambda$ are essentially constant at $k_{\text{pert}}$ and $k_{\text{M}}$ so that the dynamics of localized matter distribution remains unchanged and there is no conflict with the classical tests of general relativity. It is not then difficult to show that system (57) with the scaling (56) has the following solution:

对于“非宇宙学”标度 $k \gg\ k_{\text{cosmo}}$ (例如 $k \approx k_{\text{pert}}, k_M$ ) 处 $G$ 和 $\Lambda$ 的重整化群 (RG) 流，我们假设这些标度下对 $k$ 的依赖非常弱甚至为零，因此亚宇宙学标度下可以还原为标准引力。因此，对于 $k \lesssim k_{\text{cosmo}}$ 我们假设红外不动点行为 (56) 成立。对于 $k \gtrsim k_{\text{cosmo}}$ 我们假设 $G$ 和 $\Lambda$ 仅极弱依赖于 $k$ ，或完全不依赖 $k$ 。通过这种方式，我们在小于宇宙学标度 $\propto t$ 的长度尺度上，还原出 $G, \Lambda =$ 为常数的标准引力。具体来说，$G$ 和 $\Lambda$ 在 $k_{\text{pert}}$ 和 $k_{\text{M}}$ 处基本为常数，因此局域物质分布的动力学保持不变，与广义相对论的经典检验没有冲突。不难证明，带有标度关系 (56) 的方程组 (57) 存在下述解:

$$a(t) = \left[\left(\frac{3}{8}\right)^2 (1+w)^4 g_* \lambda_* \mathcal{M}\right]^{1/(3+3w)} t^{4/(3+3w)} \tag{62}$$

$$\rho(t) = \frac{8}{9\pi(1+w)^4 g_* \lambda_*} \frac{1}{t^4} \tag{63}$$

$$G(t) = \frac{3}{8}(1+w)^2 g_* \lambda_* t^2 \tag{64}$$

$$\Lambda(t) = \frac{8}{3(1+w)^2} \frac{1}{t^2} \tag{65}$$

Apart from the parameter $w$ and the product $g_* \lambda_*$ , the solution depends only on a single constant of integration, $\mathcal{M}$ , whose value affects only the overall scale of $a(t)$ . Numerically, it equals $8\pi\rho(t)\,[a(t)]^{3+3w} \equiv \mathcal{M}$ which, like in standard cosmology, is a conserved quantity. The solution (62) has several very interesting and attractive features. Introducing the critical density

除参数 $w$ 和乘积 $g_* \lambda_*$ 外，该解仅依赖单个积分常数 $\mathcal{M}$ ，其值仅影响 $a(t)$ 的整体标度。数值上它等于 $8\pi\rho(t)\,[a(t)]^{3+3w} \equiv \mathcal{M}$ ，与标准宇宙学中一样，这是一个守恒量。式 (62) 的解有若干非常引人关注的优良性质。引入临界密度

$$\rho_{\text{crit}}(t) \equiv \frac{3}{8\pi G(t)}\left(\frac{\dot{a}}{a}\right)^2, \tag{66}$$

we find for any value of $w, g_* \lambda_*$ , and $\mathcal{M}$ that $\rho_{\text{crit}}(t) = 2\rho(t)$ and $\rho_\Lambda(t) = \rho(t)$ . Hence,

对任意 $w, g_* \lambda_*$ 和 $\mathcal{M}$ 的值，我们可得 $\rho_{\text{crit}}(t) = 2\rho(t)$ 和 $\rho_\Lambda(t) = \rho(t)$ 。因此，

$$\rho = \rho_\Lambda = \frac{1}{2}\rho_{\text{crit}} \tag{67}$$

Thus, the total energy density $\rho_{\text{tot}} \equiv \rho + \rho_\Lambda$ equals precisely the critical one: $\rho_{\text{tot}}(t) = \rho_{\text{crit}}(t)$ . This latter equality does not come as a surprise because also the RG-improved Friedmann equation can be brought to the form

由此，总能量密度 $\rho_{\text{tot}} \equiv \rho + \rho_\Lambda$ 恰好等于临界密度: $\rho_{\text{tot}}(t) = \rho_{\text{crit}}(t)$ 。后一等价并不意外，因为经重整化群改进的弗里德曼方程也可写为如下形式

$$K = \dot{a}^2\left[\rho_{\text{tot}}/\rho_{\text{crit}} - 1\right] \tag{68}$$

so that $\rho_{\text{tot}} = \rho_{\text{crit}}$ holds true for any solution with $K = 0$. On the other hand, the exact equality of the matter energy density $\rho$ and the vacuum energy density $\rho_\Lambda$ is a nontrivial prediction of the fixed-point solution. In terms of the relative densities,

因此对任意满足 $K = 0$ 的解，$\rho_{\text{tot}} = \rho_{\text{crit}}$ 都成立。另一方面，物质能量密度 $\rho$ 与真空能量密度 $\rho_\Lambda$ 严格相等是不动点解的非平凡预言。用相对密度表示即为

$$\Omega_{\text{M}} = \Omega_\Lambda = \frac{1}{2},\ \Omega_{\text{tot}} = 1. \tag{69}$$

Also the Hubble parameter of the solution

该解的哈勃参数也

$$H \equiv \frac{\dot{a}}{a} = \frac{4}{3+3w}\frac{1}{t} \tag{70}$$

and its deceleration parameter

其减速参数

$$q \equiv -\frac{a\ddot{a}}{\dot{a}^2} = \frac{3w-1}{4} \tag{71}$$

are independent of $g_*, \lambda_*$, and $\mathcal{M}$. An interesting feature of the fixed-point solution is that it yields a universal, Time-independent value of the ”Machian” quantity $\rho G t^2$ [48]

都与 $g_*, \lambda_*$ 和 $\mathcal{M}$ 无关。不动点解的一个有趣性质是，它给出了“马赫”量 $\rho G t^2$ [48] 的一个不随时间变化的普适值

$$\rho(t)\, G(t)\, t^2 = \frac{1}{3\pi(1+w)^2}. \tag{72}$$

Note that the product $G(t)\Lambda(t) = G(k)\Lambda(k) = g_*\lambda_*$ is constant in the vicinity of any fixed point. Its actual value is characteristic of this fixed point. While, for pure gravity, $g_*^{\text{UV}}\lambda_*^{\text{UV}} = O(1)$ at the UV fixed point, the hypothetical IR fixed point of the coupled gravity-matter system has $g_*^{\text{IR}}\lambda_*^{\text{IR}} = O\left(10^{-120}\right)$. It is important to understand that the smallness of this number does not pose any fine-tuning problem as in the standard situation. In fact, in our approach both $g_*^{\text{UV}}\lambda_*^{\text{UV}}$ and $g_*^{\text{IR}}\lambda_*^{\text{IR}}$ are fixed and well-defined numbers which, at least in principle, can be computed from the RG equation. However, apart from being a difficult task technically, their actual determination is possible only once we know the complete system of all matter fields in the universe. The number $10^{-120}$ reflects specific properties of this matter system coupled to gravity rather than an initial condition. Assuming the existence of the IR fixed point and the validity of the equations (57), we are led to conclude that the late universe, for which the RG trajectory is already sufficiently close to the fixed point, is described by the power laws (62). This leads to the unambiguous prediction that $\Omega_{\text{M}} = \Omega_\Lambda = 1/2$ for every value of $w$. Moreover, if we make the additional assumption that the late universe is matter dominated $(w = 0)$, Eq. (71) yields $a \propto t^{4/3}$ with the deceleration parameter $-1/4$. Hence, near the fixed point,

请注意，乘积 $G(t)\Lambda(t) = G(k)\Lambda(k) = g_*\lambda_*$ 在任意不动点附近都是常数，其实际值是该不动点的特征。对于纯引力，$g_*^{\mathrm{UV}}\lambda_*^{\mathrm{UV}} = O(1)$ 位于紫外不动点，而假设存在的引力-物质耦合系统的红外不动点满足 $g_*^{\mathrm{IR}}\lambda_*^{\mathrm{IR}} = O\left(10^{-120}\right)$。需要明确的是，该数值很小并不像标准情况那样会引发任何精细调节问题: 实际上，在我们的研究框架中，$g_*^{\mathrm{UV}}\lambda_*^{\mathrm{UV}}$ 和 $g_*^{\mathrm{IR}}\lambda_*^{\mathrm{IR}}$ 都是固定且定义明确的数值，至少在原理上可以通过重整化群方程计算得出。然而，除了技术上难度很高之外，只有当我们知晓宇宙中所有物质场的完整系统后，才能实际确定这些数值。数值 $10^{-120}$ 反映的是物质系统与引力耦合的特定性质，而非初始条件。假设红外不动点存在且方程 (57) 成立，我们可以得出结论: 对于重整化群轨迹已经足够接近不动点的晚期宇宙，其行为由幂律 (62) 描述。由此可以得到明确预言: 对任意 $w$ 取值都满足 $\Omega_{\mathrm{M}} = \Omega_\Lambda = 1/2$ 。此外，如果我们额外假设晚期宇宙由物质主导 $(w = 0)$ ，由式 (71) 可得 $a \propto t^{4/3}$ ，其中减速参数为 $-1/4$ 。因此，在不动点附近，

$$\Omega_{\mathrm{M}} = \Omega_\Lambda = \frac{1}{2},\ q = -\frac{1}{4}\ \ (w = 0) \tag{73}$$

and therefore, the fixed-point structure provides a natural explanation for the mysterious equality (or approximate equality) of $\rho$ and $\rho_\Lambda$ . This success supports the idea that the present-day universe is in the, or at least close to the, IR fixed-point regime. The deviation of the observed values from $\Omega_{\mathrm{M}} = \Omega_\Lambda = 1/2$ could be due to the fact that the fixed-point behavior is not fully developed yet so that the universe still has some way to go before the finer quantitative details of the scaling solution is realized.

因此不动点结构自然解释了为何 $\rho$ 与 $\rho_\Lambda$ 会存在神秘的相等 (或近似相等) 关系。这一成果支持了当前宇宙处于或至少接近红外不动点区域的观点。观测值与 $\Omega_{\mathrm{M}} = \Omega_\Lambda = 1/2$ 存在偏差，可能是因为不动点行为尚未完全形成，宇宙在实现标度解更精确的定量细节之前仍需要演化一段时间。

A further testable prediction of the fixed-point hypothesis is the time variation of Newton's constant, so that

不动点假设的另一个可检验预言是牛顿常数随时间变化，因此

$$\frac{\dot{G}}{G} = \frac{2}{t} = \frac{1}{2}(3 + 3w)H(t). \tag{74}$$

The experimental upper bound from laboratory and solar system experiments for the present-day value of this quantity [49] is of the order of $|\dot{G}/G| \lesssim \left(10^{-13}\mathrm{yr}\right)^{-1}$ . Hence, even the technology available today is not very far away from being able to verify or falsify (74). One should bear in mind, however, that the $G$ in Eq. (74) refers to a different length scale than the one measured in solar system experiments, say. We also emphasize that the standard experimental value of Newton's constant, $G_{\exp}$ , does not coincide with the value $G(k = \xi/t_0)$ which is relevant for cosmology today, i.e., for $t = t_0$ . $G_{\exp}$ is measured (today) at $k_{\exp} \propto \ell^{-1}$ where the length $\ell \equiv \ell_{\mathrm{sol}}$ is a typical solar system length scale, say. Thus, in terms of the running Newton constant, $G_{\exp} = G(k = \xi'/\ell_{\mathrm{sol}})$ , since $\ell_{\mathrm{sol}} \ll t_0$ , and since in the presence of several scales, the relevant cutoff is always the larger one It is only the cosmological quantity $G(k = \xi/t)$ which grows $\propto t^2$ in the fixed-point regime, not $G_{\exp}$ . This remark entails that a $t^2$ -growth of the cosmological Newton constant in the recent past does not ruin the predictions about primordial nucleosynthesis which requires that $G(k = \xi/t_{\mathrm{nucl}})$ coincides with $G_{\exp}$ rather precisely. In fact, at the time $t = t_{\mathrm{nucl}}$ of nucleosynthesis, the cosmological Newton constant was indeed $G(k = \xi/t_{\mathrm{nucl}}) \approx G_{\exp}$ since $ct_{\mathrm{nucl}}$ and $\ell_{\mathrm{sol}}$ are of the same order of magnitude (a few light minutes.)

实验室与太阳系实验给出的该物理量当前值的实验上限 [49] 约为 $|\dot{G}/G| \lesssim \left(10^{-13}\text{yr}\right)^{-1}$ 。因此，即便以现有技术，距离验证或证伪式 (74) 也并不遥远。但需注意，式 (74) 中的 $G$ 所对应的长度尺度，不同于 (例如) 太阳系实验中测得的长度尺度。我们还需要强调，牛顿常数的标准实验值 $G_{\text{exp}}$ ，与当前宇宙学相关的即对应 $t = t_0$ 的值 $G(k = \xi/t_0)$ 并不一致。$G_{\text{exp}}$ 是当今在 $k_{\text{exp}} \propto \ell^{-1}$ 尺度测得的，其中 $\ell \equiv \ell_{\text{sol}}$ 可以说是典型的太阳系长度尺度。因此，就跑动牛顿常数 $G_{\text{exp}} = G(k = \xi'/\ell_{\text{sol}})$ 而言，由于 $\ell_{\text{sol}} \ll t_0$ ，且在存在多个尺度时，相关截断始终是更大的那个尺度: 只有宇宙学物理量 $G(k = \xi/t)$ 在不动点区域会随 $\propto t^2$ 增长，$G_{\text{exp}}$ 不会。这一结论意味着，近代宇宙学牛顿常数的 $t^2$ 增长不会破坏原初核合成的预言——原初核合成要求 $G(k = \xi/t_{\text{nucl}})$ 与 $G_{\text{exp}}$ 精确符合。实际上，在核合成发生的时刻 $t = t_{\text{nucl}}$ ，由于 $ct_{\text{nucl}}$ 和 $\ell_{\text{sol}}$ 量级相同 (均为几光分)，当时宇宙学牛顿常数确实就是 $G(k = \xi/t_{\text{nucl}}) \approx G_{\text{exp}}$ 。

The idea of the IRFP scaling has been further extended and tested in several investigations. In particular in [50] a complete transition from standard FRW to the IRFP scaling has been confronted with the available observational data. More recent works have instead implemented the IRFP hypothesis within a Swiss cheese (Einstein-Strauss) model of the universe significantly extending the original IRFP to the case of an induced running cosmological constant due to the contribution of a uniform distribution of local antigravity sources [51-54].

IRFP 标度的思想已经在多项研究中得到进一步拓展与检验。特别是文献 [50] 中，标准 FRW 宇宙到 IRFP 标度的完整演化已经和现有观测数据进行了比对。而近期的研究则将 IRFP 假设应用于瑞士奶酪 (爱因斯坦-施特劳斯) 宇宙模型中，把原始 IRFP 大幅推广到了由均匀分布的局域反引力源贡献产生的感应跑动宇宙学常数的情形 [51-54]。

## Primordial Density Fluctuations

## 原初密度涨落

Most of the approaches discussed so far are based on the renormalization group improvement, either at the level of field equations or at the level of the action. This strategy has the advantage of being easily implementable in various contexts, but its applicability when several competing scales are present in the problem is ambiguous. In most cases RG improvement can only be trusted to extract qualitative information on the dynamical properties of the NGFP, for instance, its antiscreening character. However, a question remains: is it possible to directly test the presence of the NGFP in cosmological data?

迄今为止讨论的大多数方法都基于重整化群改进，无论是在场方程层面还是作用量层面。该策略的优势是可轻松在多种场景下实现，但当问题中存在多个相互竞争的标度时，其适用性并不明确。大多数情况下，我们只能依靠重整化群改进来获取 NGFP 动力学性质的定性信息，例如它的反屏蔽特性。但仍有一个问题: 能否直接在宇宙学观测数据中检验 NGFP 的存在?

It is believed that the structure formation in the universe started out from primordial density fluctuations $\delta\rho(\mathbf{x})$ which were triggered by quantum mechanical fluctuations. As the universe expanded, those density fluctuations got amplified and magnified and finally gave rise to the large-scale structures which we observe today. This idea has been worked out in the framework of inflationary cosmology. The basic idea is to consider linear quantum fluctuations of the scalar field and the metric around a classical background:

目前普遍认为，宇宙结构形成起源于由量子涨落触发的原初密度涨落 $\delta\rho(\mathbf{x})$。随着宇宙膨胀，这些密度涨落不断被放大，最终形成了我们如今观测到的宇宙大尺度结构。该理论是在暴胀宇宙学的框架下发展出来的，基本思路是研究经典背景下标量场和度规的线性量子涨落:

$$\phi(t,\mathbf{x}) = \overline{\phi}(t) + \delta\widehat{\phi}(t,\mathbf{x}),\ g_{\mu\nu} = a^2(t)\left(\eta_{\mu\nu} + \widehat{h}_{\mu\nu}(t,\mathbf{x})\right). \tag{75}$$

If density perturbations only are of interest, then it is possible to circumvent the complications of quantizing gravity: one can consider the quantization of the scalar field and use semiclassical method of stochastic inflation at wavelengths greater than the Hubble radius. On the other hand, in order to describe the background of gravitational radiation, it is essential to properly quantize the gravitational field because the zero-point gravitational fluctuations contribute to the observed CMB temperature anisotropy.

如果仅关注密度涨落，就可以绕过引力量子化的复杂性: 我们可以只对标量场进行量子化，在大于哈勃半径的波长上使用随机暴胀的半经典方法。另一方面，要描述引力辐射的背景，必须对引力场进行适当量子化，因为零点引力涨落会对观测到的 CMB 温度各向异性产生贡献。

Within the AS scenario the quantization of the gravitational field does not pose any conceptual difficulty, and therefore, it is possible to imagine an alternative scenario for which primordial density fluctuations were generated already during the Planck era as the aftermath of the big bang. In fact, due to the running of the Newton's constant, the background evolution solves the horizon problems without the need of an inflationary field, as the past light cones are broadening for $t < t_{\text{class}}$ where $t_{\text{class}}$ is a classical time after which the running of Newton’s constant to the presence of the NGFP can be neglected and the geometry is classical (see [55] for further details). Obviously the most natural assumption about the quantum origin of $\delta\rho$ is that, before $t \approx t_{\text{class}}$, the quantum fluctuations of the metric itself generated the primordial density fluctuations by some decoherence mechanism. As it is possible to show, this assumption naturally leads to an almost scale-free (Harrison-Zeldovich) fluctuation spectrum. In fact the two-point correlation function

在渐近安全 (AS) 框架中，引力场量子化不存在概念层面的困难，因此可以提出一个替代场景: 原初密度涨落早在普朗克时代就作为大爆炸的产物产生了。实际上，由于牛顿常数的跑动，背景演化可以在不需要暴胀场的情况下解决视界问题，因为对于 $t < t_{\text{class}}$，过去光锥会不断拓宽，其中 $t_{\text{class}}$ 是一个经典时间，在该时间之后，牛顿常数向 NGFP 存在的跑动可以忽略，几何退化为经典几何 (进一步细节参见文献 [55])。显然，关于 $\delta\rho$ 的量子起源最自然的假设是，在 $t \approx t_{\text{class}}$ 之前，度规本身的量子涨落通过某种退相干机制产生了原初密度涨落。可以证明，该假设自然得出几乎无标度的 (哈里森-泽尔多维奇) 涨落谱。实际上，两点关联函数

$$\xi(\mathbf{x}) = \langle \delta(\mathbf{x}+\mathbf{y})\,\delta(\mathbf{y}) \rangle \tag{76}$$

of the density contrast $\delta(\mathbf{x}) \equiv \delta\rho(\mathbf{x})/\langle\rho\rangle_t$ at some fixed time $t \lesssim t_{\text{class}}$ close to the end of the Planck era when the spectrum is "handed over" from the quantum gravity to the classical regime. One defines the power spectrum by

是密度反差 $\delta(\mathbf{x}) \equiv \delta\rho(\mathbf{x})/\langle\rho\rangle_t$ 在普朗克时代末期附近固定时刻 $t \lesssim t_{\text{class}}$ 的关联函数，此时涨落谱正好从量子引力区域过渡到经典区域。我们可以通过下式定义功率谱:

$$|\delta_k|^2 \equiv V\int d^3x\xi(\mathbf{x})\,e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{77}$$

so that the fluctuation spectrum has the spectral index $n$ if $|\delta_k|^2$ has the form of a power law $|\delta_k|^2 \propto |\mathbf{k}|^n$ . (V denotes the normalization volume.) On a flat background, the effective graviton propagator for the fixed-point regime is proportional to $\widetilde{\mathcal{G}}(p) \propto 1/p^4$ which amounts to $\mathcal{G}(x,y) \propto \ln(x-y)^2$ in position space. This form of the propagator is valid for $p^2 \gg m_{\rm Pl}^2$ or $(x-y)^2 \ll \ell_{\rm Pl}^2$ , respectively. The logarithmic two-point function may be understood as a limiting case of the familiar "critical" propagator $\mathcal{G}(x,y) \propto 1/|x-y|^{d-2+\eta}$ for $d=4$ and the anomalous dimension $\eta \equiv \eta_N(g_*,\lambda_*) = -2$ which characterizes the UV fixed point. Let us look at the curvature fluctuation $\delta\mathbf{R} \propto \partial\partial h$ caused by a fluctuation $h_{\mu\upsilon}(x)$ of the metric. (Here a symbolic notation where $\mathbf{R}$ stands for the curvature scalar or for any component of the Riemann or Einstein tensor is implied.) Because $\langle h_{\mu\upsilon}(x)\,h_{\lambda\tau}(y)\rangle \propto \ln(x-y)^2$ , the curvature correlation function is $\langle\delta\mathbf{R}(x)\,\delta\mathbf{R}(y)\rangle \propto 1/(x-y)^4$ , rather than $\propto 1/(x-y)^6$ as implied by the tree-level propagator. Therefore, the leading short-distance singularity in a curved space-time is given by $\langle\delta\mathbf{R}(x)\,\delta\mathbf{R}(y)\rangle \propto 1/d(x,y)^4$ where $d(x,y)$ is the geodesic distance of $x$ and $y$ . This formula is applicable when the space-time curvature is small compared to $1/d(x,y)^2$ .

因此若 $|\delta_k|^2$ 满足幂律形式 $|\delta_k|^2 \propto |\mathbf{k}|^n$，涨落谱的谱指数为 $n$。(V 代表归一化体积。) 在平直背景下，不动点 regime 的有效引力子传播子正比于 $\widetilde{\mathcal{G}}(p) \propto 1/p^4$， 对应位置空间中的 $\mathcal{G}(x,y) \propto \ln(x-y)^2$ 。该传播子形式分别适用于 $p^2 \gg m_{\rm Pl}^2$ 或 $(x-y)^2 \ll \ell_{\rm Pl}^2$ 的情况。对数两点关联函数可以理解为常见的“临界”传播子 $\mathcal{G}(x,y) \propto 1/|x-y|^{d-2+\eta}$ 在 $d=4$ 与表征紫外不动点的反常维数 $\eta \equiv \eta_N(g_*,\lambda_*) = -2$ 下的极限情况。下面我们来看由度规涨落 $h_{\mu\upsilon}(x)$ 引起的曲率涨落 $\delta\mathbf{R} \propto \partial\partial h$ 。(这里采用符号约定，$\mathbf{R}$ 代表曲率标量，或是黎曼张量或爱因斯坦张量的任意分量。) 由于 $\langle h_{\mu\upsilon}(x)\,h_{\lambda\tau}(y)\rangle \propto \ln(x-y)^2$，曲率关联函数为 $\langle\delta\mathbf{R}(x)\,\delta\mathbf{R}(y)\rangle \propto 1/(x-y)^4$， 而非树级传播子给出的 $\propto 1/(x-y)^6$ 。因此，弯曲时空领头阶短距离奇点由 $\langle\delta\mathbf{R}(x)\,\delta\mathbf{R}(y)\rangle \propto 1/d(x,y)^4$ 给出，其中 $d(x,y)$ 是 $x$ 和 $y$ 之间的测地线距离。该公式适用于时空曲率远小于 $1/d(x,y)^2$ 的情况。

Let us consider the background of a Robertson-Walker space-time and we put $x$ and $y$ on the same time slice. Hence, $d(x,y) = a(t)\,|\mathbf{x}-\mathbf{y}|$ where $\mathbf{x}$ and $\mathbf{y}$ are the comoving Cartesian coordinates of $x$ and $y$ , respectively. This leads to

我们考虑罗伯逊-沃尔克时空背景，将 $x$ 和 $y$ 取在同一时间切片上，因此可得 $d(x,y) = a(t)\,|\mathbf{x}-\mathbf{y}|$，其中 $\mathbf{x}$ 和 $\mathbf{y}$ 分别为 $x$ 和 $y$ 的共动笛卡尔坐标。由此可得

$$\langle\delta\mathbf{R}(\mathbf{x},t)\,\delta\mathbf{R}(\mathbf{y},t)\rangle \propto \frac{1}{|\mathbf{x}-\mathbf{y}|^4}. \tag{78}$$

In the scenario where the primordial density fluctuations are generated by quantum fluctuations, one assumes that the classical statistical expectation value (76) is proportional to a quantum mechanical expectation value $\langle\Psi\left|\widehat{\phi}(\mathbf{x}+\mathbf{y})\,\widehat{\phi}(\mathbf{y})\right|\Psi\rangle$ where $\widehat{\phi}$ is the operator whose fluctuations are supposed to become classical. In the case at hand where we assume that $\delta\rho$ originates from the fluctuations of the spacetime geometry Itself, the natural choice for $\widehat{\phi}$ is $\widehat{\phi} \propto \mathbf{R}$ , i.e., to some extent an arbitrary linear combination of curvature components. In fact, already classically the Einstein equations imply $8\pi G\delta\rho = -\delta G_0{}^0$ where $G_\mu{}^\upsilon$ is the Einstein tensor. As a consequence, the two-point function of $\widehat{\phi}$ is proportional to the $\delta\mathbf{R}$ correlator (78). Therefore, the correlation function of $\delta\rho$ behaves as

在原初密度涨落由量子涨落产生的模型中，我们假设经典统计期望值 (76) 正比于量子力学期望值 $\langle\Psi\left|\hat{\phi}(\mathbf{x}+\mathbf{y})\hat{\phi}(\mathbf{y})\right|\Psi\rangle$，其中 $\hat{\phi}$ 是涨落最终演化为经典的算符。就我们目前假设 $\delta\rho$ 起源于时空几何本身的涨落而言，$\hat{\phi}$ 的自然选择是 $\hat{\phi} \propto \mathbf{R}$，即它在一定程度上是曲率分量的任意线性组合。事实上，即使在经典层面，爱因斯坦方程也给出 $8\pi G\delta\rho = -\delta G_0{}^0$，其中 $G_\mu{}^\nu$ 是爱因斯坦张量。由此可得，$\hat{\phi}$ 的两点函数正比于 $\delta\mathbf{R}$ 关联函数 (78)，因此 $\delta\rho$ 的关联函数满足:

$$\xi(\mathbf{x}) \propto \frac{1}{|\mathbf{x}|^4} \tag{79}$$

provided the physical distance $a(t)|\mathbf{x}|$ is smaller than $\ell_{\mathrm{Pl}}$. The power spectrum of the modes with physical momenta $|\mathbf{k}|/a(t) \lesssim m_{\mathrm{Pl}}$ (at fixed time $t \lesssim t_{\text{class}}$) is given by the three-dimensional Fourier transform of (79):

前提是物理距离 $a(t)|\mathbf{x}|$ 小于 $\ell_{\mathrm{Pl}}$。物理动量为 $|\mathbf{k}|/a(t) \lesssim m_{\mathrm{Pl}}$ (固定时刻为 $t \lesssim t_{\text{class}}$) 的模式的功率谱可由 (79) 的三维傅里叶变换给出:

$$|\delta_k|^2 \propto |\mathbf{k}| \ . \tag{80}$$

This is precisely the Harrison-Zeldovich scale-invariant spectrum with the spectral index $n = 1$. We can thus imagine that "sub-Hubble scale" modes evolve according to the standard theory of cosmological perturbations starting with a scale-invariant spectrum immediately after the quantum gravity epoch, $t \gtrsim t_{\mathrm{Pl}}$.

这正是谱指数为 $n = 1$ 的哈里森-泽尔多维奇标度不变谱。因此我们可以认为，“哈勃视界以下尺度”模式按照宇宙扰动标准理论演化，在量子引力阶段 $t \gtrsim t_{\mathrm{Pl}}$ 结束之初就已经具有标度不变谱。

Deviations from the value $n = 1$ are expected if the generation of primordial perturbations does not occur precisely at the NGFP and in this case a more complete treatment which would include both the fluctuations from the background and the fluctuations around the background in a consistent way is needed. Recent calculations [56, 57] based on a direct calculation of the graviton propagator around a flat space have computed the spectral function of the transverse-traceless mode of the graviton. The resulting propagator can be recast as the propagator in the form of a diffeomorphism effective action

如果原初扰动并非精确产生于非高斯定点 (NGFP)，那么谱指数就会偏离 $n = 1$，此时需要更完整的处理，将背景涨落和围绕背景的涨落一致地纳入考虑。近期基于直接计算平坦空间周围引力子传播子的研究 [56, 57] 已经算出引力子横向无迹模的谱函数。得到的传播子可以改写为微分同胚有效作用量形式的传播子

$$S = \int d^4x\sqrt{g}\left(\frac{R}{16\pi G_{\mathrm{N}}} + C_{\mu\nu\rho\sigma} f_C(\Box)\, C^{\mu\nu\rho\sigma}\right), \tag{81}$$

where the term $R_{\mu\nu\rho\sigma} f_R(\Box)\, R^{\mu\nu\rho\sigma}$ has been neglected being of the same order of the curvature expansion and which relates to the physical scalar mode of the graviton. It is hoped that recent progress in the calculation of the form factors in quantum gravity [58] will eventually provide us with ab initio calculations of the graviton spectrum in the early universe in a consistent way, by taking into account the possible contribution from the matter sector.

其中 $R_{\mu\nu\rho\sigma} f_R(\Box) R^{\mu\nu\rho\sigma}$ 项因和曲率展开同阶已被忽略，该项与引力子的物理标量模相关。我们希望，随着量子引力中形状因子计算的最新进展 [58]，未来能通过考虑物质部分的可能贡献，以一致的方式从第一性原理得到早期宇宙中引力子谱的计算结果。

## Conclusions

## 结论

The consequences of the asymptotically safe scenario in quantum gravity are still under intense investigation. In its simplest implementation the running of $G$ is encoded in a new set of scale-dependent effective field equations. An important nontrivial result is represented by the possibility of describing our universe as an evolution from a state of zero entropy where all the observed entropy is indeed produced from the running of $G$ and $\Lambda$ , and the classical era naturally arises from the presence of the Gaussian fixed point. It is remarkable that this result emerges only from the running of $G$ and $\Lambda$ according to the $\beta$ -functions (10). Moreover, in the presence of matter bouncing non-singular cosmologies are obtained from the antiscreening behavior of $G$ at high densities. Those models represent a physically viable alternative to the inflation scenario but further work should be done to discuss structure formation in these models. On the other hand, if inflation is correct and its scale is still not too far from the energy scale of the ”basin of attraction” of the NGFP, then it is possible to determine a class of $f(R)$ -corrections to standard inflationary models which, hopefully, could be tested in future CMB experiments. The most promising candidate from this point of view is the modified Starobinsky model described in [35] where a $R^{3/2}$ modification of the standard $R + R^2$ inflation could provide a significantly higher (and therefore testable) value for the tensor-to-scalar ratio $r$ . Albeit a direct probe of the existence of the NGFP is still not available, recent progress in the calculation of the graviton spectrum in the Lorentzian theory could in principle offer the possibility to determine the initial power spectrum of the cosmological tensor perturbation in a fully consistent way. It is hoped that future CMB anisotropy experiments like LiteBIRD [59] could shed new important light on the presence of the NGFP.

渐近安全情景在量子引力中的后果仍处于深入研究阶段。在其最简实现中，$G$ 的跑动由一组新的依赖于标度的有效场方程描述。一个重要的非平凡结果是，我们可以将宇宙描述为从零熵状态演化而来，所有观测到的熵确实由 $G$ 和 $\Lambda$ 的跑动产生，经典时期由高斯不动点的存在自然演化而来。值得注意的是，该结果仅由遵循 $\beta$ 函数 (10) 的 $G$ 和 $\Lambda$ 的跑动得出。此外，当存在物质时，$G$ 在高密度下的反屏蔽行为可以得到非奇异的反弹宇宙学。这些模型是暴胀情景在物理上可行的替代方案，但仍需进一步研究来讨论这些模型中的结构形成问题。另一方面，如果暴胀理论正确，且其能标与 NGFP 的“吸引域”能标相差不远，那么我们可以得到一类对标准暴胀模型的 $f(R)$ 修正，有望在未来的 CMB 实验中得到检验。从这一角度来看，最有前景的候选者是文献 [35] 中描述的修正斯塔罗宾斯基模型，其中对标准 $R + R^2$ 暴胀的 $R^{3/2}$ 修正可以给出张量-标度比 $r$ 更高 (因而可检验) 的取值。尽管目前仍无法直接探测 NGFP 的存在，洛伦兹理论中引力子谱计算的最新进展原则上可以为我们提供机会，以完全自洽的方式确定宇宙学张量扰动的初始功率谱。我们希望未来的 CMB 各向异性实验，例如 LiteBIRD[59]，能够为 NGFP 是否存在提供新的重要线索。

## Cross-References

## 交叉引用

Form Factors in Asymptotically Safe Quantum Gravity

渐近安全量子引力中的形状因子

The Functional f(R) Approximation

泛函 f(R) 近似

The Functional Renormalization Group in Quantum Gravity

量子引力中的泛函重整化群

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